

THICK-THIN DECOMPOSITION OF FLOER TRAJECTORIES AND ADIABATIC GLUING

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ABSTRACT. This is a sequel to [OZ1] in which we studied the adiabatic degeneration of Floer trajectories to “disk-flow-disk” configurations and the recovering gluing, where the gradient flow part had length 0. In the present paper we study the case when the gradient flow trajectory has a positive length. Unlike the standard gluing problem, we study the problem of gluing two objects of different dimensions: 1-dimensional gradient segments and 2-dimensional (perturbed) J -holomorphic maps. We assume that the join points of the J -holomorphic curve to the gradient segments are immersed. The immersion condition at the joint points of the “disk-flow-disk” configurations plays crucial role in the proof of surjectivity of our gluing, and is further illustrated by an explicit example in \mathbb{CP}^n .

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1. INTRODUCTION

Let (M, ω) be a compact symplectic manifold.

This is a sequel to [OZ1] in which we study the adiabatic degeneration of maps $u : \mathbb{R} \times S^1 \rightarrow M$ satisfying the following 1-parameter ($0 < \varepsilon < \varepsilon_0$) family of Floer equations:

$$(du + P_{K_\varepsilon}(u))_{J_\varepsilon}^{(0,1)} = 0 \quad \text{or equivalently} \quad \bar{\partial}_{J_\varepsilon}(u) + (P_{K_\varepsilon})_{J_\varepsilon}^{(0,1)}(u) = 0, \quad (1.1)$$

We refer to section 3 for detailed description of the one-parameter family Hamiltonian K_ε and almost complex structure J_ε , and the equation (1.1), the invariant form of the Floer equation. The expression of the degenerating Hamiltonian $K_\varepsilon : \mathbb{R} \times S^1 \times M \rightarrow \mathbb{R}$ has the form

$$K_\varepsilon(\tau, t, x) = \begin{cases} \kappa_\varepsilon^+(\tau) \cdot H(t, x) & \text{for } \tau \geq R(\varepsilon) \\ \kappa_\varepsilon^0(\tau) \cdot \varepsilon f(x) & \text{for } |\tau| \leq R(\varepsilon) \\ \kappa_\varepsilon^-(\tau) \cdot H(t, x) & \text{for } \tau \leq -R(\varepsilon) \end{cases} \quad (1.2)$$

where κ_ε^\pm and κ_ε^0 are suitable cut-off functions (See subsection 3 for the precise definition.)

Roughly speaking, the adiabatic degeneration occurs because K_ε restricts to Morse function εf on longer and longer cylinder $[-R(\varepsilon), R(\varepsilon)] \times S^1$ in $\mathbb{R} \times S^1$. A basic assumption that we had put on [OZ1] was that $R(\varepsilon)$ satisfies

$$\lim_{\varepsilon \rightarrow 0} \varepsilon R(\varepsilon) = 0. \quad (1.3)$$

The main purpose of the present paper is to prove a gluing result when we have the *non-zero* limit

$$\lim_{\varepsilon \rightarrow 0} \varepsilon R(\varepsilon) = \ell$$

for $\ell > 0$ as promised in [OZ1]. Under this assumption, it is proved in [Oh3] and [MT] that as $\varepsilon \rightarrow 0$, a degenerating sequence of Floer trajectories converges to a “*disk-flow-disk*” configuration denoted by (u_-, χ, u_+) where u_\pm satisfy the equation similar to (1.1) with K_ε is replaced by K_\pm defined by

$$K_+ = \begin{cases} 0 & \text{near } o_+ \\ H_+(t, x) dt & \text{near } e_+ \end{cases} \quad (1.4)$$

and

$$K_- = \begin{cases} 0 & \text{near } o_- \\ H_-(t, x) dt & \text{near } e_- \end{cases} \quad (1.5)$$

Here $H_\pm : S^1 \times M \rightarrow \mathbb{R}$ is a Hamiltonian function independent of the variable τ , and χ is a gradient trajectory of f that satisfies $\dot{\chi} + \text{grad}f(\chi) = 0$ on $[0, \ell]$ such that

$$u_-(o_-) = \chi(-l), \quad \chi(l) = u_+(o_+). \quad (1.6)$$

Let $\dot{\Sigma}_+$ be the Riemann sphere with one marked point o_+ and one positive puncture e_+ . Choose analytical charts at o_+ and at e_+ on some neighborhoods O_+ and E_+ respectively, so that conformally $O_+ \setminus o_+ \cong (-\infty, 0] \times S^1$, and $E_+ \setminus e_+ \cong [0, +\infty) \times S^1$. We use t for the S^1 coordinate and τ for the \mathbb{R} coordinate. Then $\{-\infty\} \times S^1$ and $\{+\infty\} \times S^1$ correspond to o_+ and e_+ respectively.

Let $z_\pm : S^1 \rightarrow M$ be a nondegenerate periodic orbit of H_\pm and consider a finite energy solution $u_\pm : \dot{\Sigma} \rightarrow M$ of the Floer equation (1.4), (1.5) associated to K_\pm respectively. By the finite energy condition and since $K_\pm \equiv 0$ near o_\pm , u_\pm extend smoothly across o_\pm and can be regarded as a smooth map defined on \mathbb{C} that is holomorphic near the origin $0 \in \mathbb{C}$.

Now we consider the lifting $[z_\pm, w_\pm]$ of z_\pm and introduce the main moduli spaces of our interest in a precise term.

Let J_\pm be a pair of domain-dependent almost complex structures on M and let $u_\pm : \mathbb{R} \times S^1 \rightarrow M$ be solutions of (1.1) with K_ε replaced by K_\pm given in (1.4), (1.5) in class $A_\pm \in \pi_2(z_\pm)$ with a marked point $o_\pm \in S^2$ respectively, and $\chi : [-l, l] \rightarrow M$ is a gradient segment of the Morse function f connecting the two points $u_+(o_+)$ and $u_-(o_-)$. We assume (J_\pm) satisfy $J_\pm \equiv J_0$ near the punctures o_\pm respectively and generic in that u_\pm are Fredholm regular. We also assume that the pair (J_0, f) is generic in that χ are Fredholm regular and the configuration (u_-, χ, u_+) satisfies the “disc-flow-disc” transversality defined in [OZ1] Proposition 5.2. We will recall this definition below. Define the moduli space

$$\begin{aligned} \mathcal{M}(K_\pm, J_\pm; z_\pm; A_\pm) &= \left\{ (u_\pm, o_\pm) \mid u_\pm : S^2 \setminus \{e_\pm, o_\pm\} \rightarrow M \right. \\ &\quad \left. \text{satisfying (1.4), (1.5) respectively} \right\} \end{aligned}$$

and the evaluation map

$$ev_\pm : \mathcal{M}(K_\pm, J_\pm; z_\pm; A_\pm) \rightarrow M, \quad u_\pm \rightarrow ev_\pm(o_\pm).$$

Definition 1.1. The configuration (u_-, χ, u_+) satisfies the “disc-flow-disc” transversality if the map

$$\begin{aligned} \phi_f^{2l} \circ ev_- \times ev_+ & : \quad \mathcal{M}(K_-, J_-; z_-; A_-) \times \mathcal{M}(K_+, J_+; z_+; A_+) \rightarrow M \times M \\ (u_-, u_+) & \rightarrow \left(\phi_f^{2l} u_-(o_-), u_+(o_+) \right) \end{aligned}$$

is transversal to the diagonal $\Delta \subset M \times M$, where ϕ_f^{2l} is the time- $2l$ flow of the Morse function f .

The “disc-flow-disc” transversality can be achieved by a generic choice of the pair (J_0, f) (Corollary 6.3).

We note that for each given pair $(A_-, A_+) \in \pi_2(z_-) \times \pi_2(z_+)$ we consider and fix a class $B \in \pi_2(z_-, z_+)$ such that

$$A_- \# B \# A_+ = 0 \quad \text{in } \pi_2(M). \quad (1.7)$$

(Such $B \in \pi_2(z_-, z_+)$ is unique modulo the action of π_1). We define

$$\mathcal{M}^{dfd}(K_-, J_-; f; K_+, J_+; B)$$

to be the fiber product

$$\mathcal{M}(K_-, J_-; z_-; A_-) \phi_f^l \circ_{ev_-} \times_{ev_+} \mathcal{M}(K_+, J_+; z_+; A_+). \quad (1.8)$$

We denote by

$$\mathcal{M}^\varepsilon := \mathcal{M}(K_\varepsilon, J_\varepsilon; z_-, z_+; B)$$

the moduli space of solutions to (1.1) and consider

$$\mathcal{M}_{(0, \varepsilon_0]}^{para} := \bigcup_{0 < \varepsilon \leq \varepsilon_0} \mathcal{M}^\varepsilon.$$

In this paper, we prove the following main theorem:

Theorem 1.2. $\mathcal{M}_{(0, \varepsilon_0]}^{para}$ can be embedded into a manifold with boundary $\overline{\mathcal{M}}_{[0, \varepsilon_0]}^{para}$ whose boundary is diffeomorphic to $\mathcal{M}(K_-, J_-; f; K_+, J_+; B)$, i.e., there is a diffeomorphism

$$Glue : [0, \varepsilon_0) \times \mathcal{M}^{dfd}(K_-, J_-; f; K_+, J_+; B) \rightarrow \overline{\mathcal{M}}_{[0, \varepsilon_0]}^{para}$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_{(0, \varepsilon_0]}^{para} & \hookrightarrow & \overline{\mathcal{M}}_{[0, \varepsilon_0]}^{para} \xrightarrow{Glue^{-1}} [0, \varepsilon_0] \times \mathcal{M}^{dfd}(K_-, J_-; f; K_+, J_+; A_- \# A_+) \\ \downarrow & & \downarrow \\ (0, \varepsilon_0) & \hookrightarrow & [0, \varepsilon_0] \end{array}$$

The statement of this theorem is the analog to the corresponding theorem, Theorem 10.19 [OZ1] for the case $\ell = 0$. However the gluing analysis and its off-shell framework are of nature very different from those of [OZ1]: In [OZ1], we need to *rescale the target* near the nodal point of nodal Floer trajectories, while in the present paper we do not need to rescale the target but we need to *renormalize* the domain variable (τ, t) to $(\tau/\varepsilon, t/\varepsilon)$ to obtain the limiting gradient flow of f arising from the thin part of the degeneration of \mathcal{M}^ε as $\varepsilon \rightarrow 0$ as done in [FO1]. The length ℓ of the limiting gradient trajectory is determined by the limit $\lim_{\varepsilon \rightarrow 0} \varepsilon R(\varepsilon) = \ell$, i.e., by how fast the length of thin part of degenerating Floer trajectories increases relative to the speed as $\varepsilon \rightarrow 0$. We do not need to rescale the target M unlike the case in [OZ1]. This is because the presence of gradient line with *non-zero* length carries enough information to recover all the solutions in \mathcal{M}^ε with *immersed join points* nearby the elements from $\mathcal{M}(K_- |f| K_+; A_- \# A_+)$ in Gromov-Hausdorff topology, as long as ε is sufficiently small. On the other hand, as $\ell = 0$ the Sobolev constant in present paper blows up which obstructs application of the same scheme which led us to blowing up target appears to be necessary as in [OZ1]. *In both cases, we need to assume that the Floer trajectories either at the nodal points or at the join point of the gradient trajectories are immersed.* It appears that without this assumption gluing analysis is much more delicate. The immersion condition can be achieved for a generic Floer datum under the small index conditions (Proposition 6.5) which naturally enter in the construction of various operations in Floer theory.

Incidentally we would like to point out that the main theorem can be also used to give another proof of isomorphism property of the PSS map which uses a piecewise

smooth cobordism different from that of [OZ1]. Namely we directly go from “disk-flow-disk” configurations to resolved Floer trajectories by the above mentioned adiabatic gluing theorem *without passing through the nodal Floer trajectories* unlike the one proposed in [PSS] and realized in [OZ1]. Unlike the case of [OZ1], we do not need to change the target this time. We also remark that the full gradient trajectory (i.e. $l = \infty$) case can be treated by the same method of this paper, which is in fact easier, because we do not need to put the power weight $\rho(\tau)$ on χ (see Remark 5.7).

Some remarks on the relationship with other relevant literature are now in order.

We would like to point out that our gluing analysis in the present paper and in [OZ1] can be applied to many other contexts where similar thick-thin decomposition occurs (including $l = 0$ case) during the adiabatic degeneration arises. One of the very first instances where importance of this kind of gluing analysis was mentioned is the papers [Fu], [Oh1]. An analogous analysis was also carried out by Fukaya and the senior author in [FO1] when there is no non-constant bubbles around.

The gluing analysis with non-constant bubbles around for the equation

$$\frac{\partial u}{\partial \tau} + J_0 \left(\frac{\partial u}{\partial t} - X_{\varepsilon f}(u) \right) = 0$$

as $\varepsilon > 0$ small was pursued by Fukaya and the senior author at that time but left unfinished [FO2] since then, partly because the analysis turned out to be much more nontrivial than they originally expected. The gluing analysis given in the present paper can be similarly applied also to this case *under the assumption that the join points are immersed*. Section 15 of the present paper briefly outlines the necessary modification for this case in the setting of both Hamiltonian and Lagrangian contexts. One outcome of this analysis and the dimension counting argument proves the following equivalence theorem which was expected in [Oh1]. We state the theorem only for the case where we do not need any additional argument except some immediate translations of our gluing analysis presented in this paper without using the virtual machinery.

Theorem 1.3. *Let (M, ω) be a monotone symplectic manifold and $L \subset (M, \omega)$ be a monotone Lagrangian submanifold. Fix a Darboux neighborhood U of L and identify U with a neighborhood of the zero section in T^*L . Consider $k + 1$ Hamiltonian deformations of L by autonomous Hamiltonian functions $F_0, \dots, F_k : M \rightarrow \mathbb{R}$ such that*

$$F_i = \chi f_i \circ \pi$$

where $f_0, f_1, \dots, f_k : L \rightarrow \mathbb{R}$ are generic Morse functions, χ is a cut-off function such that $\chi = 1$ on U and supported nearby U . Now consider

$$L_{i,\varepsilon} = \text{Graph}(\varepsilon df_i) \subset U \subset M, \quad i = 0, \dots, k.$$

Assume transversality of L_i 's of the type given in [FO1]. Consider the intersections $p_i \in L_i \cap L_{i+1}$ such that

$$\dim \mathcal{M}(L_0, \dots, L_k; p_0, \dots, p_k) = 0, 1.$$

Then when ε is sufficiently small, the moduli space $\mathcal{M}(L_{0,\varepsilon}, \dots, L_{k,\varepsilon}; p_0, \dots, p_k)$ is diffeomorphic to the moduli space of pearl complex defined in [Oh1, Oh2], [BC].

The following is an immediate corollary of this theorem.

Corollary 1.4. *Under the same hypothesis on (M, ω) and L , the A_∞ -structure proposed in [Fu, Oh1, Oh2], which also coincides with the quantum structure defined in [BC], is isomorphic to the A_∞ -structure defined in [FOOO].*

In [BO], Bourgeois and Oancea studied the Floer equation of autonomous Hamiltonian H under the assumption that its 1-periodic orbits are transversally nondegenerate on a symplectic manifold with contact type boundary, in relation to the study of the linearized contact homology of a fillable contact manifold and symplectic homology of its filling. Their construction is based on Morse-Bott techniques for Floer trajectories where a function of the type εf is used on the Morse-Bott set of *parameterized* periodic orbits of H to break the S^1 symmetry of the orbit space, whose loci consist of *isolated, unparameterized, nonconstant* periodic orbits. The Hamiltonian pearl complex case treated in subsection 15.1 can be put in a similar Morse-Bott setting this time to break the M -symmetry of constant orbits of Hamiltonian 0 by adding a small Morse function εf on M itself.

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2. INVARIANT SET-UP OF THE FLOER EQUATION

In this section, we formulate the set-up for the general Floer's perturbed Cauchy-Riemann equation on compact Riemann surface with a finite number of punctures. This requires a coordinate-free framework of the equation.

2.1. Punctures with analytic coordinates. We start with the description of positive and negative *punctures*. Let Σ be a compact Riemann surface with a marked point $p \in \Sigma$. Consider the corresponding punctured Riemann surface $\dot{\Sigma}$ with an analytic coordinates $z : D \setminus \{p\} \rightarrow \mathbb{C}$ on a neighborhood $D \setminus \{p\} \subset \dot{\Sigma}$. By composing z with a linear translation of \mathbb{C} , we may assume $z(p) = 0$.

We know that $D \setminus \{p\}$ is conformally isomorphic to both $[0, \infty) \times S^1$ and $(-\infty, 0] \times S^1$.

- (1) We say that the pair $(p; (D, z))$ has a *incoming cylindrical end* (with analytic chart) if we have

$$D = z^{-1}(D^2(1))$$

and are given by the biholomorphism

$$(\tau, t) \in S^1 \times (-\infty, 0] \mapsto e^{2\pi(\tau+it)} \in D^2(1) \setminus \{0\} \mapsto z^{-1} \in D \setminus \{p\}.$$

We call the corresponding puncture $p \in \Sigma$ a *positive puncture*.

- (2) We say that the pair $(p; (D, z))$ has a *outgoing cylindrical end* (with analytic chart) if we have

$$D = z^{-1}(D^2(1))$$

and are given by the biholomorphism

$$(\tau, t) \in S^1 \times [0, \infty) \mapsto e^{-2\pi(\tau+it)} \in D^2(1) \setminus \{0\} \mapsto z^{-1} \in D \setminus \{p\}.$$

In this case, we call the corresponding puncture $(p; (D, z))$ a *negative puncture* (with analytic chart).

2.2. Hamiltonian perturbations. Now we describe the Hamiltonian perturbations in a coordinate free fashion.

Let Σ be a compact Riemann surface and $\dot{\Sigma}$ denote Σ with a finite number of punctures and analytic coordinates. We denote by $\mathcal{J}_{0,\omega}$ the set of almost complex structures that are cylindrical near the puncture with respect to the given analytic charts $z = e^{\pm(2\pi(\tau+it))}$. Define \mathcal{J}_{Σ} or $\dot{\mathcal{J}}_{\Sigma}$ to be the set of maps $J : \Sigma, \dot{\Sigma} \rightarrow \mathcal{J}_{0,\omega}$ respectively.

Definition 2.1. We call $K \in \Omega^1(\Sigma, C^\infty(M))$ *cylindrical* at the puncture $p \in \Sigma$ with analytic chart (D, z) , if it has the form

$$K(\tau, t) = H(t) dt$$

in $D \setminus \{p\}$. We denote by $\mathcal{K}_{\dot{\Sigma}}$ the set of such K 's.

One important quantity associated to the one-form K is a two-form, denoted by R_K , and defined by

$$R_K(\xi_1, \xi_2) = \xi_1[K(\xi_2)] - \xi_2[K(\xi_1)] - \{K(\xi_2), K(\xi_1)\} \quad (2.1)$$

for two vector fields ξ_1, ξ_2 , where $\xi_1[K(\xi_2)]$ denotes directional derivative of the function $K(\xi_2)(z, x)$ with respect to the vector field ξ_1 as a function on Σ , holding the variable $x \in M$ fixed. It follows from the expression that R_K is tensorial on Σ .

The Hamiltonian-perturbed Cauchy-Riemann equation has the form

$$(du + P_K(u))_J^{(0,1)} = 0 \quad \text{or equivalently} \quad \bar{\partial}_J(u) + (P_K)_J^{(0,1)}(u) = 0 \quad (2.2)$$

on Σ in general.

For each given such pair (K, J) , it defines a perturbed Cauchy-Riemann operator by

$$\bar{\partial}_{(K,J)}u := \bar{\partial}_Ju + P_K(u)_J^{(0,1)} = (du + P_K(u))_J^{(0,1)}.$$

Let (\mathbf{p}, \mathbf{q}) be a given set of positive punctures $\mathbf{p} = \{\mathbf{p}_1, \dots, \mathbf{p}_\ell\}$ and with negative punctures $\mathbf{q} = \{\mathbf{q}_1, \dots, \mathbf{q}_\ell\}$ on Σ . For each given Floer datum (K, J) and a collection $\vec{z} = \{z_*\}_{* \in \mathbf{p} \cup \mathbf{q}}$ of asymptotic periodic orbits z_* attached to the punctures $* = p_i$ or $* = q_j$, we consider the perturbed Cauchy-Riemann equation

$$\begin{cases} \bar{\partial}_{(K,J)}(u) = 0 \\ u(\infty_*, t) = z_*(t). \end{cases} \quad (2.3)$$

Our main interest will lie in the case where $(|\mathbf{p}|, |\mathbf{q}|)$ is either $(1, 0)$, $(0, 1)$ or $(1, 1)$ in the present paper.

One more ingredient we need to give the definition of the Hamiltonian-perturbed moduli space is the choice of an appropriate energy of the map u . For this purpose, we fix a metric h_Σ which is compatible with the structure of the Riemann surface and which has the cylindrical ends with respect to the given cylindrical coordinates near the punctures, i.e., h_Σ has the form

$$h_\Sigma = d\tau^2 + dt^2 \quad (2.4)$$

on $D_* \setminus \{*\}$. We denote by dA_Σ the corresponding area element on Σ .

Here is the relevant energy function

Definition 2.2 (Energy). For a given asymptotically cylindrical pair (K, J) , we define

$$E_{(K,J)}(u) = \frac{1}{2} \int_{\Sigma} |du - P_K(u)|_J^2 dA_{\Sigma}$$

where $|\cdot|_J$ is the norm of $\Lambda^{(0,1)}(u^*TM) \rightarrow \Sigma$ induced by the metrics h_{Σ} and $g_J := \omega(\cdot, J\cdot)$.

Note that this energy depends only on the conformal class of h_{Σ} , i.e., depends only on the complex structure j of Σ and restricts to the standard energy for the usual Floer trajectory moduli space given by

$$E_{(H,J)} = \frac{1}{2} \int_{C_*} \left(\left| \frac{\partial u}{\partial \tau} \right|_J^2 + \left| \frac{\partial u}{\partial t} - X_H(u) \right|_J^2 \right) dt d\tau$$

in the cylindrical coordinates (τ, t) on the cylinder C_* corresponding to the puncture $*$. $E_{(K,J)}(u)$ can be bounded by a more topological quantity depending only on the asymptotic orbits, or more precisely their liftings to the *universal covering space* of $\mathcal{L}_0(M)$, where the latter is the contractible loop space of M . As usual, we denote such a lifting of a periodic orbit z by $[z, w]$ where $w : D^2 \rightarrow M$ is a disc bounding the loop z .

We also consider the *real blow-up* of $\dot{\Sigma} \subset \Sigma$ at the punctures and denote it by $\overline{\Sigma}$ which is a compact Riemann surface with boundary

$$\partial \overline{\Sigma} = \coprod_{* \in \mathfrak{p} \cup \mathfrak{q}} S_*^1$$

where S_*^1 is the exceptional circle over the point $*$. We note that since there is given a preferred coordinates near the point $*$, each circle S_*^1 has the canonical identification

$$\theta_* : S_*^1 \rightarrow \mathbb{R}/\mathbb{Z} = [0, 1] \mod 1.$$

We note that for a given asymptotic orbits \vec{z} , one can define the space of maps $u : \dot{\Sigma} \rightarrow M$ which can be extended to $\overline{\Sigma}$ such that $u \circ \theta_* = z_*(t)$ for $* \in \mathfrak{p} \cup \mathfrak{q}$. Each such map defines a natural homotopy class B relative to the boundary. We denote the corresponding set of homotopy classes by $\pi_2(\vec{z})$. When we are given the additional data of bounding discs w_* for each z_* , then we can form a natural homology (in fact a homotopy class), denoted by $B\# \left(\coprod_{* \in \mathfrak{p} \cup \mathfrak{q}} [w_*] \right) \in H_2(M)$, by ‘capping-off’ the boundary components of B using the discs w_* respectively.

Definition 2.3. Let $\{[z_*, w_*]\}_{* \in \mathfrak{p} \cup \mathfrak{q}}$ be given. We say $B \in \pi_2(\vec{z})$ is *admissible* if it satisfies

$$B\# \left(\coprod_{* \in \mathfrak{p} \cup \mathfrak{q}} [w_*] \right) = 0 \quad \text{in } H_2(M, \mathbb{Z}) \quad (2.5)$$

where

$$\# : \pi_2(\vec{z}) \times \prod_{* \in \mathfrak{p} \cup \mathfrak{q}} \pi_2(z_*) \rightarrow H_2(M, \mathbb{Z})$$

is the natural gluing operation of the homotopy class from $\pi_2(\vec{z})$ and those from $\pi_2(z_*)$ for $* \in \mathfrak{p} \cup \mathfrak{q}$. Now we are ready to give the definition of the Floer moduli spaces.

Definition 2.4. Let (K, J) be a Floer datum over Σ with punctures $\mathfrak{p}, \mathfrak{q}$, and let $\{[z_*, w_*]\}_{* \in \mathfrak{p} \cup \mathfrak{q}}$ be the given asymptotic orbits. Let $B \in \pi_2(\tilde{z})$ be a homotopy class admissible to $\{[z_*, w_*]\}_{* \in \mathfrak{p} \cup \mathfrak{q}}$. We define the moduli space

$$\mathcal{M}(K, J; \{[z_*, w_*]\}_*) = \{u : \dot{\Sigma} \rightarrow M \mid u \text{ satisfies (2.3) and } [u] \# (\coprod_{* \in \mathfrak{p} \cup \mathfrak{q}} [w_*]) = 0\}. \quad (2.6)$$

We note that the moduli space $\mathcal{M}(K, J; \{[z_*, w_*]\}_*)$ is a finite union of the moduli spaces

$$\mathcal{M}(K, J; \tilde{z}; B); \quad B \# \left(\prod_{* \in \mathfrak{p} \cup \mathfrak{q}} [w_*] \right) = 0.$$

3. ADIABATIC FAMILY OF FLOER MODULI SPACES

In this section, we give the precise description of one-parameter family of perturbed Cauchy-Riemann equations in a coordinate-free form parameterized by $\varepsilon > 0$ such that as $\varepsilon \rightarrow 0$ the equation becomes degenerate in a suitable sense which we will make precise.

We first consider the cases $\mathfrak{p}_\pm = \{e_\pm\}$, $\mathfrak{q}_\pm = \{o_\pm\}$ with one negative and one positive punctures. Then we consider the one-form K_\pm such that

$$K_\pm(\tau, t, x) = \kappa^\pm(\tau) H_t(x) \quad (3.1)$$

on U_\pm in terms of the given analytic coordinates.

Now we consider a one-parameter family $(K_\varepsilon, J_\varepsilon)$ with $R = R(\varepsilon) \rightarrow \infty$,

$$\varepsilon R(\varepsilon) \rightarrow \ell \quad (3.2)$$

with $\ell \geq 0$ as $\varepsilon \rightarrow 0$. More precise description of $(K_\varepsilon, J_\varepsilon)$ is in order.

To define this family, we fix any two continuous functions $R(\varepsilon)$ and $S(\varepsilon)$ that satisfies

$$\varepsilon R(\varepsilon) \rightarrow \ell, \quad \varepsilon S(\varepsilon) \rightarrow 0 \quad (3.3)$$

as $\varepsilon \rightarrow 0$. For example, we can choose

$$R(\varepsilon) = \frac{\ell}{\varepsilon}, \quad S(\varepsilon) = \frac{1}{2\pi} \ln(1 + \frac{\ell}{\varepsilon}) \quad (3.4)$$

and introduce

$$\tau(\varepsilon) = R(\varepsilon) + \frac{p-1}{\delta} S(\varepsilon)$$

for the convenience of our exposition later, which will appear frequently in our calculations. Here we fix any $p > 2$ and $0 < \delta < 1$. The $p > 2$ is for $W^{1,p} \hookrightarrow C^0$ Sobolev embedding on Riemann surface, and $0 < \delta < 1$ is to get rid of the 0-spectrum of $i \frac{\partial}{\partial t}$ on S^1 . Any such choice of p and δ will be suffice for our analysis. The choice of $\tau(\varepsilon)$ is not canonical as it depends on p and δ , but when $p \rightarrow 2$ and $\delta \rightarrow 1$, $|\tau(\varepsilon) - R(\varepsilon)|$ is close to $S(\varepsilon)$.

Then we decompose \mathbb{R} into

$$-\infty < -\tau(\varepsilon) - 1 < -\tau(\varepsilon) < -R(\varepsilon) < R(\varepsilon) < \tau(\varepsilon) < \tau(\varepsilon) + 1 < \infty.$$

Let $\dot{\Sigma}_\pm$ be two compact surfaces each with two positive puncture (resp. one negative puncture) with analytic coordinates. Let $e_\pm, o_\pm \in \dot{\Sigma}_\pm$ two marked points and denote by (τ, t) with $\pm\tau \geq 0$ the cylindrical charts of $\dot{\Sigma}_\pm \setminus \{o_\pm\}$ such that $z = e^{\pm 2\pi(\tau + it)}$.

We choose neighborhoods U_{\pm} of e_{\pm} and analytic charts $\varphi_{\pm} : U_{\pm} \rightarrow \mathbb{C}$ and the associated coordinates $z = e^{2\pi(\tau+it)}$ so that

$$\varphi_+(U_+ \setminus \{e_+\}) \cong [0, \infty) \times S^1, \quad \varphi_-(U_- \setminus \{e_-\}) \cong (-\infty, 0] \times S^1,$$

we fix a function

$$\kappa^+(\tau) = \begin{cases} 0 & 0 \leq \tau \leq 1 \\ 1 & \tau \geq 2 \end{cases} \quad (3.5)$$

and let $\kappa^-(\tau) = \kappa^+(-\tau)$.

We define a glued surface with a cylindrical coordinates, denoted by

$$\dot{\Sigma} := \Sigma_+ \# \Sigma_-$$

Then $\dot{\Sigma}$ carries two natural marked points (e_-, e_+) . We pick an embedded path passing a point $o \in \Sigma_+ \cap \Sigma_-$ and connecting e_-, e_+ . We fix the unique cylindrical coordinates $\dot{\Sigma} \cong \mathbb{R} \times S^1$ mapping the path to $\mathbb{R} \times \{0\}$ and o to $(0, 0)$.

With this cylindrical coordinates, we consider $\varepsilon > 0$ and a family of cut-off functions defined by $\kappa_{\varepsilon}^+(\tau) = \kappa^+(\tau - \tau(\varepsilon) + 1)$ and $\kappa_{\varepsilon}^-(\tau) = \kappa_{\varepsilon}^+(-\tau)$. It is easy to see

$$\kappa_{\varepsilon}^+(\tau) = \begin{cases} 1 & \text{for } \tau \geq \tau(\varepsilon) + 1 \\ 0 & \text{for } 0 \leq \tau \leq \tau(\varepsilon) \end{cases}, \quad \kappa_{\varepsilon}^- = \begin{cases} 1 & \text{for } \tau \leq -\tau(\varepsilon) - 1 \\ 0 & \text{for } \tau(\varepsilon) \leq \tau \leq 0 \end{cases} \quad (3.6)$$

We then define $(K_{\varepsilon}, J_{\varepsilon})$ to be the glued family

$$K_{\varepsilon}(\tau, t) = \begin{cases} \kappa_{\varepsilon}^+(\tau) \cdot H_t & \tau \geq R(\varepsilon) \\ \kappa_{\varepsilon}^0(\tau) \cdot \varepsilon f & |\tau| \leq R(\varepsilon) \\ \kappa_{\varepsilon}^-(\tau) \cdot H_t & \tau \leq -R(\varepsilon). \end{cases} \quad (3.7)$$

$$J_{\varepsilon}^{\pm}(\tau, t, x) = \begin{cases} J^{\kappa_{\varepsilon}^{\pm}(\tau)}(t, x) & \tau \geq R(\varepsilon) \\ J_0(x) & |\tau| \leq R(\varepsilon) \\ J^{\kappa_{\varepsilon}^-(\tau)}(t, x) & \tau \leq -R(\varepsilon) \end{cases} \quad (3.8)$$

associated to $\kappa_{\varepsilon}^{\pm}$ respectively, where $\kappa_{\varepsilon}^0(\tau)$ is a smooth cut-off function such that

$$\kappa_{\varepsilon}^0(\tau) = \begin{cases} 1 & \text{for } |\tau| \leq R(\varepsilon) \\ 0 & \text{for } |\tau| \geq R(\varepsilon) + 1, \end{cases} \quad (3.9)$$

$$|\kappa_{\varepsilon}^0(\tau)| \leq 1, \quad |(\kappa_{\varepsilon}^0)'(\tau)| \leq 2.$$

We will vary $R = R(\varepsilon)$ depending on ε and study the family of equation

$$(du + P_{K_{\varepsilon}}(u))_J^{(0,1)} = 0 \quad (3.10)$$

as $\varepsilon \rightarrow 0$.

4. ADIABATIC CONVERGENCE

By definition of K_{ε} and J_{ε} , as $\varepsilon \rightarrow 0$, on the domain

$$[-R(\varepsilon), R(\varepsilon)] \times S^1$$

we have $K_{\varepsilon}(\tau, t) \equiv \varepsilon f$ and $J_R(\tau, t) \equiv J_0$, and so (1.1) becomes

$$\frac{\partial u}{\partial \tau} + J_0 \left(\frac{\partial u}{\partial t} - \varepsilon X_f(u) \right) = 0.$$

Furthermore $K_\varepsilon^\pm(\tau, t) \equiv H_t dt$, $J_\varepsilon^\pm(\tau, t) \equiv J_t$ on

$$\mathbb{R} \times S^1 \setminus [-\tau(\varepsilon) - 1, \tau(\varepsilon) + 1] \times S^1$$

(3.10) is cylindrical at infinity, i.e., invariant under the translation in τ -direction at infinity.

Note that on any fixed compact set $B \subset \mathbb{R} \times S^1$, we will have

$$B \subset [-R(\varepsilon), R(\varepsilon)] \times S^1$$

for all sufficiently small ε . And as $\varepsilon \rightarrow 0$, $K_\varepsilon \rightarrow 0$ on B in C^∞ -topology, and hence the equation (3.10) converges to $\bar{\partial}_{J_0} u = 0$ on B in that $J \rightarrow J_0$ and $K_\varepsilon \rightarrow 0$ in C^∞ -topology. On the other hand, after translating the region $(-\infty, -(R(\varepsilon) - \frac{1}{3})]$ to the right (resp. $[R(\varepsilon) - \frac{1}{3}, \infty)$ to the left) by $2R(\varepsilon) - \frac{2}{3}$ in τ -direction, (3.10) converges to

$$\frac{\partial u}{\partial \tau} + J_\varepsilon^+ \left(\frac{\partial u}{\partial t} - X_H(u) \right) = 0$$

on $(-\infty, 0] \times S^1$ (resp. on $[0, \infty) \times S^1$) and $\bar{\partial}_{J_0} u = 0$ on $[0, R(\varepsilon) - \frac{1}{3}] \times S^1$ (resp. on $[-R(\varepsilon) + \frac{1}{3}, 0] \times S^1$).

Now we are ready to state the meaning of the *adiabatic convergence* for a sequence u_n of solutions $(du_n + P_{K_{\varepsilon_n}})_{J_{\varepsilon_n}}^{(0,1)} = 0$ as $n \rightarrow \infty$. After taking away bubbles, we can assume that we have the derivative bound

$$|du_n| < C < \infty \quad (4.1)$$

where we take the norm $|du_n|$ with respect to the given metric g on M .

From now on, we will always assume that we have this derivative bound, i.e., that bubble does not occur as $\varepsilon \rightarrow 0$.

We denote

$$\Theta_\varepsilon = [-R(\varepsilon), R(\varepsilon)] \times S^1$$

and consider the local energy

Definition 4.1. We define the *local energy* of u on B by

$$E_{J,B}(u) := \int_B |du|_J^2 dt d\tau.$$

There are two cases to consider :

- (1) there exists a subsequence n_i and $c > 0$ such that $E_{J, \Theta_{\varepsilon_{n_i}}}(u_{n_i}) > c > 0$ for all sufficiently large i ,
- (2) $\limsup_{n \rightarrow \infty} E_{J, \Theta_{\varepsilon_n}}(u_n) = 0$.

For the case (1), standard argument produces a non-constant J_0 -holomorphic sphere. To describe the situation (2) in a precise manner, we introduce the following notion of adiabatic convergence to a disc-flow-disc trajectory (u_-, χ, u_+) . This would describe a neighborhood basis of (u_-, χ, u_+) in a suitable completion of $\mathcal{F}^{1,p}(K_{\varepsilon_n}, J_{\varepsilon_n}; z_-, z_+; B)$.

We first recall the definition of Hausdorff distance of subsets $A, B \subset X$

$$d_H(A, B) = \sup_{x \in A} d(x, B) + \sup_{y \in B} d(A, y).$$

4.1. Adiabatic deformations of domain. To provide a rigorous definition of the adiabatic convergence, one has to give a precise way of degenerating the punctured sphere equipped with the cylindrical structure at the punctures to the union of two spheres joined by a line segment of length 2ℓ parameterized by the interval $[-\ell, \ell]$.

For $U_{\pm} \subset \Sigma_{\pm}$, in terms of analytic charts $\varphi_{\pm} : U_{\pm} \rightarrow D^2 \subset \mathbb{C}$ and the associated coordinates z , we identify $U_{\pm} \setminus \{o_{\pm}\}$ with the open punctured unit disc $D^2 \setminus \{0\}$. We first consider the annular domain of $U_{\pm} \setminus \{o_{\pm}\}$:

$$\text{Ann}_{\alpha} := \{z \in D^2 \mid |\alpha|^{3/4} \leq |z| \leq |\alpha|^{1/4}\} = \{z \in D^2 \mid |R_{\alpha}|^{-3/2} \leq |z| \leq |R_{\alpha}|^{-1/2}\}.$$

The choice of the power is dictated by that of Fukaya-Ono's deformation given in [FO] in which $|\alpha| = R_{\alpha}^{-2}$ for $|\alpha|$ sufficiently small.

We note that the conformal modulus of Ann_{α} is $\|\alpha\|^{-1/2} = R_{\alpha}$. For each given $\varepsilon > 0$, we choose $\alpha(\varepsilon)$ so that

$$\frac{p-1}{\delta} S(\varepsilon) = \ln \|\alpha(\varepsilon)\|^{-1/2}.$$

We recall the choice of $S(\varepsilon)$

$$S(\varepsilon) = \frac{1}{2\pi} \ln \left(1 + \frac{\ell}{\varepsilon} \right).$$

Then we choose a biholomorphism

$$\varphi_{\varepsilon}^{-} : \text{Ann}_{\alpha(\varepsilon)} \rightarrow [-\tau(\varepsilon), -R(\varepsilon)] \times S^1.$$

Similarly we define

$$\varphi_{\varepsilon}^{+} : \text{Ann}_{\alpha(\varepsilon)} \rightarrow [R(\varepsilon), \tau(\varepsilon)] \times S^1.$$

Using this, we define a family of glued surfaces

$$\Sigma_{\varepsilon} = \left(\Sigma_{-} \setminus D^2(|\alpha(\varepsilon)|^{3/4}) \right) \cup_{\varphi_{\varepsilon}^{-}} ([-\tau(\varepsilon), \tau(\varepsilon)] \times S^1) \cup_{(\varphi_{\varepsilon}^{+})^{-1}} \left(\Sigma_{+} \setminus D^2(|\alpha(\varepsilon)|^{3/4}) \right).$$

We denote

$$C(\varepsilon) = [-\tau(\varepsilon), \tau(\varepsilon)] \times S^1$$

with the standard metric $g_{C(\varepsilon)}$. Then through the identifications $\varphi_{\pm} : U_{\pm} \rightarrow D^2$, this family give rise to the following ε -parameterized family of resolved cylinders $(\Sigma_{\varepsilon}^{adi}, g_{\varepsilon}^{adi})$ equipped with the metric provided by

$$g_{\varepsilon}^{adi} = \begin{cases} g_{+} & \text{on } \Sigma_{+} \setminus U_{+}(|\alpha(\varepsilon)|) \\ g_{C(\varepsilon)} & \text{on } C(\varepsilon) \\ g_{-} & \text{on } \Sigma_{-} \setminus U_{-}(|\alpha(\varepsilon)|) \end{cases}$$

and suitably interpolated in between. Note that the conformal structure of Σ_{ε} is degenerating on the annuli region $C(\varepsilon)$ when $\varepsilon \rightarrow 0$, and any given compact subset $K \subset \Sigma_{\pm} \setminus \{o_{\pm}\}$ is covered by $\Sigma_{\pm} \setminus U_{\pm}(\delta)$ for a sufficiently small $\delta > 0$.

4.2. Definition of adiabatic convergence. Now we involve maps defined on the resolved domains $(\Sigma_{\varepsilon}^{adi}, g_{\varepsilon}^{adi})$ and provide a precise definition of adiabatic convergence or of adiabatic topology near the disc-flow-disc moduli space

$$\mathcal{M}_{\ell}^{dfd}(z_{-}; f; z_{+}; B)$$

inside the off-shell spaces

$$\overline{\mathcal{F}}^{para}(z_{-}; f; z_{+}; B) = \bigcup_{\ell \geq \ell_0} \mathcal{F}^{\ell}(z_{-}; f; z_{+}; B).$$

Definition 4.2. Let $\{\varepsilon_j\}$ be sequence with $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$.

We say a sequence u_j of maps in $\mathcal{F}^{1,p}(K_{\varepsilon_j}, J_{\varepsilon_j}; z_-, z_+; B)$ $\{\varepsilon_j\}$ -*adiabatically converges* to a disc-flow-disc trajectory (u_-, χ, u_+) if the following hold:

- (1) $\lim_{j \rightarrow \infty} E_{J, \Theta_{\varepsilon_j}}(u_j) = 0$.
- (2) $\lim_{n \rightarrow \infty} d_H(u_j([-R(\varepsilon_j), R(\varepsilon_j)] \times S^1), \chi([-l, l])) = 0$, where d_H is the Gromov-Hausdorff metric.
- (3) $u_j|_{\Sigma_{\pm} \setminus U_{\pm}(\zeta)} \rightarrow u_{\pm}$ in C^∞ for any given $0 < \zeta < 1$, or equivalently,
 $u_j(\cdot \pm \tau(\varepsilon_j), \cdot) \rightarrow u_{\pm}$ in C^∞ on any domain $\pm[\frac{1}{2\pi} \ln \zeta, \infty) \times S^1$.
- (4) $\lim_{\zeta \rightarrow 0} \lim_{j \rightarrow \infty} \text{diam} \left(u_j|_{\varphi^\pm(\text{Ann}_{\alpha(\varepsilon_j)}) \setminus \pm[\frac{1}{2\pi} \ln \zeta, \tau(\varepsilon)] \times S^1} \right) = 0$, or equivalently,
 $\lim_{\zeta \rightarrow 0} \lim_{j \rightarrow \infty} \text{diam} \left(u_j \left(\pm [R(\varepsilon_j), \tau(\varepsilon_j) + \frac{1}{2\pi} \ln \zeta] \times S^1 \right) \right) = 0$.

In fact, we can turn this adiabatic convergence into a topology by describing a neighborhood basis of the topology at $\varepsilon = 0$. For any $\varepsilon, \zeta > 0$ we define

$$\begin{aligned} & d_{adia}^{\varepsilon, \zeta}(u, (u_-, \chi, u_+)) \\ := & \max \left\{ E_{J, \Theta_\varepsilon}(u), d_H(u([-R(\varepsilon), R(\varepsilon)] \times S^1), \chi([l, l])), \right. \\ & \text{diam} \left(u_j \left(\pm \left[R(\varepsilon), \tau(\varepsilon) + \frac{1}{2\pi} \ln \zeta \right] \times S^1 \right) \right), \\ & \left. d_{C_{\Sigma_{\pm} \setminus U_{\pm}(\zeta)}^\infty}(u(\cdot - \tau(\varepsilon), u_-), d_{C_{\Sigma_{\pm} \setminus U_{\pm}(\zeta)}^\infty}(u(\cdot + \tau(\varepsilon), \cdot), u_+)) \right\} \quad (4.2) \end{aligned}$$

Then the sequence u_j $\{\varepsilon_j\}$ -*adiabatically converges* to a disc-flow-disc trajectory (u_-, χ, u_+) if and only if

$$\lim_{\zeta \rightarrow 0} \lim_{j \rightarrow \infty} d_{adia}^{\varepsilon_j, \zeta}(u_j, (u_-, \chi, u_+)) = 0.$$

We define the open set $V_{\zeta, \delta}^\varepsilon$ in $\mathcal{F}^{1,p}(K_\varepsilon, J_\varepsilon; z_-, z_+; B)$ by

$$V_{\zeta, \delta}^\varepsilon = \left\{ u \in \mathcal{F}^{1,p}(K_\varepsilon, J_\varepsilon; z_-, z_+; B) \mid d_{adia}^{\varepsilon, \zeta}(u, (u_-, \chi, u_+)) < \delta \right\}. \quad (4.3)$$

For a given sequence u_j satisfying (2), we consider the reparameterization

$$\bar{u}_j(\tau, t) = u_j \left(\frac{\tau}{\varepsilon_j}, \frac{t}{\varepsilon_j} \right)$$

on the domain $[-\varepsilon_j R(\varepsilon_j), \varepsilon_j R(\varepsilon_j)] \times \mathbb{R}/2\pi\varepsilon_j\mathbb{Z}$. A straightforward calculation shows that \bar{u}_j satisfies

$$\frac{\partial \bar{u}_j}{\partial \tau} + J_0 \left(\frac{\partial \bar{u}_j}{\partial t} - X_f(\bar{u}_j) \right) = 0$$

or equivalently

$$\frac{\partial \bar{u}_j}{\partial \tau} + J_0 \frac{\partial \bar{u}_j}{\partial t} + \text{grad}_{J_0} f(\bar{u}_j) = 0$$

on $[-\varepsilon_j R(\varepsilon_j), \varepsilon_j R(\varepsilon_j)] \times \mathbb{R}/2\pi\varepsilon_j\mathbb{Z}$. For the simplicity of notation, we will sometimes denote

$$R_j = R(\varepsilon_j).$$

The following result was proved in Part II of [Oh3]. A similar result was also obtained by Mundet i Riera and Tian. (See Theorem 1.3 [MT].)

Theorem 4.3 ([Oh3], [MT]). *Suppose*

$$\ell = \lim_{j \rightarrow \infty} \varepsilon_j R(\varepsilon_j), \quad \lim_{j \rightarrow \infty} E_{J, \Theta_{R_j}}(u_j) = 0.$$

Then there exists a subsequence, again denoted by u_j , such that the reparameterized maps \bar{u}_j satisfy the following:

- (1) *Consider the supremum*

$$\text{width } \bar{u}_j|_{[-\ell, \ell] \times S^1} := \sup_{\tau \in [-\ell, \ell]} \text{diam Im } \bar{u}_j|_{\{\tau\} \times S^1}.$$

Then $\text{width } \bar{u}_j|_{[-\ell, \ell] \times S^1} \rightarrow 0$ and in particular the center of mass of $\bar{u}_j : [-\ell, \ell] \times S^1$ defines a smooth path

$$\text{cm}(\bar{u}_j) : [-\ell, \ell] \rightarrow M$$

and we can uniquely write

$$\bar{u}_j(\tau, t) = \exp_{\text{cm}(\bar{u}_j)}(\tau) \xi_j(\tau, t)$$

so that $\int_{S^1} \xi_j(\tau, t) dt = 0$ for all $\tau \in [-\ell, \ell]$.

- (2) *The path $\text{cm}(\bar{u}_j)$ converges to a gradient trajectory $\chi : [-\ell, \ell] \rightarrow M$ satisfying $\dot{\chi} + \text{grad}_J f(\chi) = 0$ and $\xi_j \rightarrow 0$ in C^∞ -topology.*

Under the assumption $\lim_{j \rightarrow \infty} E_{J, \Theta_{R_j}}(u_j) = 0$, after taking away bubbles, on $(-\infty, K] \times S^1$ of any fixed K , the translated sequences $u_j(\cdot - \tau(\varepsilon_j), \cdot) : \mathbb{R} \times S^1 \rightarrow M$ of solutions u_j of (3.10) as above converge to $u_- : \mathbb{R} \times S^1 \rightarrow M$ that satisfies the equation

$$\frac{\partial u_-}{\partial \tau} + J \left(\frac{\partial u_-}{\partial t} - X_{H_-}(u_-) \right) = 0$$

in compact C^∞ -topology, where H_\pm are the Hamiltonians

$$H_\pm(\tau, t, x) = \kappa^\pm(\tau) H(t, x).$$

Similar statement holds for $u_j(\cdot + R(\varepsilon_j) + S(\varepsilon_j), \cdot)$ at $+\infty$.

5. FREDHOLM THEORY OF FLOER TRAJECTORIES NEAR GRADIENT SEGMENTS

In this section, we study Floer trajectories near a gradient segment χ . Since χ itself is a Floer trajectory with S^1 symmetry, transversality is hard to achieve. We will set up appropriate Banach manifold hosting χ such that χ is a transversal Floer trajectory. During this section we let J be a t -independent almost complex structure compatible with ω .

5.1. The Banach manifold set up. For the gluing purpose, we treat χ as a t -independent Floer trajectory $u_\chi : [-l, l] \times S^1 \rightarrow M$ of the equation

$$\frac{\partial u}{\partial \tau} + J(u) \left(\frac{\partial u}{\partial t} - X_f(u) \right) = 0. \quad (5.1)$$

We choose a connection such that $\nabla J = 0$, and a trivialize $u_\chi^* TM$ using the parallel transport. We denote the corresponding trivialization by $\Phi : u_\chi^* TM \rightarrow [-l, l] \times \mathbb{C}^n$ over $[-l, l]$. In this trivialization, the linearization of the above equation has the form

$$D\xi = \frac{\partial \xi}{\partial \tau} + J_0 \frac{\partial \xi}{\partial t} + A(\tau) \xi \quad (5.2)$$

for any vector field $\xi : [-l, l] \times S^1 \rightarrow \mathbb{C}^n$ where J_0 is the standard complex structure on \mathbb{C}^n , independent of (τ, t) , and

$$A =: \xi \rightarrow (\nabla_{\chi'} \Phi + \nabla_{\Phi} \nabla f(\chi(\tau))) \xi$$

is a 0-th order linear differential operator. If we change f to εf , then χ is changed to $\chi_{\varepsilon} := \chi(\varepsilon \cdot)$, Φ is changed to $\Phi_{\varepsilon} := \Phi(\varepsilon \cdot, \cdot)$, and A is changed to $A_{\varepsilon} := \varepsilon A$. So without loss of any generality, we may assume that A is as small as we want by considering the Morse function $\varepsilon_0 f$ for some small ε_0 . We denote the linearization operator of (5.1) for the function εf by D^{ε} which has the form

$$D^{\varepsilon} \xi = \frac{\partial \xi}{\partial \tau} + J_0 \frac{\partial \xi}{\partial t} + A_{\varepsilon}(\tau) \xi$$

We are going to construct a Banach manifold hosting Floer trajectories nearby u_{χ} that matches u_{\pm} , on which good estimate of the right inverse Q of D can be obtained. The idea is to use Fourier series to decompose the variation vector field ξ into the “zero mode” part and the “higher mode” part. For the zero mode, it is tied with the Fredholm theory of the linearized gradient operator L of Morse function f that

$$L\xi = \frac{\partial \xi}{\partial \tau} + A(\tau) \xi.$$

For the higher mode part, it has no zero spectrum so uniform bound of the right inverse can be obtained. We remark that only the boundary value of the “zero mode” is compatible with the “disc-flow-disc” transversality defined in definition 1.1 (also in [OZ1]), while that of the “higher modes” is not needed thanks to its smaller norms compared to that of zero modes.

Remark 5.1. It is important to assume J to be t -independent hence after trivialization of $\chi^* TM \rightarrow [-l, l]$, the resulted J_0 is (τ, t) -independent. Otherwise the Fourier decomposition of the above linearized Floer equation (5.2) with respect to $t \in S^1$ becomes difficult. On the other hand, if we use t -dependent J , the transversality of χ as a Floer trajectory is easier to achieve. This is the case for the transformed Floer equation (15.5) for Lagrangian pearl complex and will be discussed later.

We define

$$\mathcal{B}_{\chi} = \{u \in W^{1,p}([-l, l] \times S^1, M) \mid \text{dist}(u(\tau, t), \chi(\tau)) < d \text{ for all } \tau, t.\}, \quad (5.3)$$

where $d > 0$ is a small constant that for any loop $w : S^1 \rightarrow (M, g)$ whose image has diameter less than d , the center of mass of the loop w is well defined, and is denoted by $\text{cm}(w)$. Therefore for any $u \in \mathcal{B}_{\chi}$, its *center of mass curve*

$$\text{cm}(u) : [-l, l] \rightarrow M$$

is well defined and is close to $\chi(\tau)$. Here

$$\tau \mapsto \int_{S^1} u(\tau, t) dt$$

is the *center of mass* of the loops $t \mapsto u(\tau, t)$, and is well defined for u close enough to χ : The center of mass of a loop $\gamma : S^1 \rightarrow M$ is defined to be the unique point $m_{\gamma} \in M$ such that

$$\int_0^1 \text{dist}^2(m, \gamma(t)) dt$$

is the minimum. (See [K] for the detailed exposition on the center of mass in general).

Remark 5.2. The center of mass of a curve in a Riemannian manifold is well defined whenever the diameter of the curve is sufficiently small. In particular the condition $\text{diam}(u(\tau, t)) < C\varepsilon$ enables us to define the center of mass of the curve $u(\tau, t)$ ($t \in S^1$) when ε is sufficiently small. Therefore we can also use a Darboux chart containing the image of $u(\tau, t)$ and may identify the curve $t \mapsto u(\tau, t)$ one in \mathbb{C}^n . With this understood, we will sometimes denote the center of mass $\bar{u}(0)$ just as the average $\int_{S^1} u(\tau, t) dt$.

Along the gradient flow $\chi(\tau)$, for any section $\xi \in \Gamma(W^{1,p}(u_\chi^* TM))$, we let the average vector field to be

$$\xi_0(\tau) := \int_{S^1} \xi(\tau, t) dt \quad (5.4)$$

and let the reduced vector field be

$$\tilde{\xi}(\tau, t) := \xi(\tau, t) - \xi_0(\tau). \quad (5.5)$$

Then we define the Banach norm of $\xi \in T_{u_\chi} \mathcal{B}_\chi$ to be

$$\|\xi\|_{(W_\rho^{1,p}([-l, l]) \times S^1)} = \|\xi_0\|_{W^{1,p}([-l, l])} + \|\tilde{\xi}\|_{W_\rho^{1,p}([-l, l]) \times S^1}, \quad (5.6)$$

where

$$\begin{aligned} \|\xi_0\|_{W^{1,p}([-l, l])}^p &= \int_{-l}^l (|\xi_0|^p + |\nabla_\tau \xi_0|^p) d\tau, \\ \|\tilde{\xi}\|_{W_\rho^{1,p}([-l, l]) \times S^1}^p &= \int_{S^1} \int_{-l}^l \left(|\tilde{\xi}|^p + |\nabla \tilde{\xi}|^p \right) (1 + |\tau|)^\delta d\tau dt \end{aligned}$$

We call $\rho(\tau) := (1 + |\tau|)^\delta$ the weighting function for the above weighted $W_\rho^{1,p}$ norm. We remark that the norm $\|\tilde{\xi}\|_{W_\rho^{1,p}([-l, l]) \times S^1}$ is equivalent to the norm $\|(1 + |\tau|)^{\frac{\delta}{p}} \tilde{\xi}\|_{W^{1,p}([-l, l]) \times S^1}$.

Similarly for any section $\eta \in \Gamma(W^{1,p}(u_\chi^* TM) \otimes_J \Lambda^{0,1}([-l, l] \times S^1))$, we let

$$\begin{aligned} \eta_0(\tau) &:= \int_{S^1} \eta(\tau, t) dt, \\ \tilde{\eta}(\tau, t) &:= \eta(\tau, t) - \eta_0(\tau). \end{aligned}$$

and define

$$\begin{aligned} \|\eta_0\|_{L^p([-l, l])}^p &= \int_{-l}^l |\eta_0|^p d\tau, \\ \|\tilde{\eta}\|_{L_\rho^p([-l, l]) \times S^1}^p &= \left\| (1 + |\tau|)^{\frac{\delta}{p}} \tilde{\eta}(\tau, t) \right\|_{L^p([-l, l]) \times S^1}^p \\ \|\eta\|_{L_\rho^p([-l, l]) \times S^1} &= \|\eta_0\|_{L^p([-l, l])} + \|\tilde{\eta}\|_{L_\rho^p([-l, l]) \times S^1} \end{aligned}$$

Along general elements $u \in \mathcal{B}_\chi$ that are close enough to χ , (which are “thin” cylinders), we define the norm $\|\xi\|_{W_\rho^{1,p}}$ for $\xi \in W^{1,p}(\Gamma(u^* TM))$ as the following: let

$$\bar{\xi}(\tau, t) = \text{Pal}_{\text{cm}(u)}^u \xi(\tau, t),$$

where $Pal_{cm(u)}^u$ is the parallel transport from $u(\tau, t)$ to $cm(u)(\tau)$ along the shortest geodesic. Then along $cm(u)$, similar to $\chi(\tau)$ case, we decompose

$$\bar{\xi}(\tau, t) = (\bar{\xi})_0(\tau) + \tilde{\xi}(\tau, t)$$

and pull back to define

$$\xi_0 = Pal_u^{cm(u)}(\bar{\xi})_0, \quad \tilde{\xi} = Pal_u^{cm(u)}\tilde{\xi}$$

where $Pal_u^{cm(u)}$ is the parallel transport from $cm(u)(\tau)$ to back $u(\tau, t)$ along the shortest geodesic. Therefore we have well defined decomposition

$$\xi(\tau, t) = \xi_0(\tau) + \tilde{\xi}(\tau, t).$$

for $W^{1,p}(\Gamma(u^*TM))$. Then we define

$$\begin{aligned} \|\xi_0\|_{W^{1,p}([-l, l])} &= \|(\bar{\xi})_0\|_{W^{1,p}([-l, l])} \\ \|\tilde{\xi}\|_{W_\rho^{1,p}([-l, l] \times S^1)} &= \|\tilde{\xi}\|_{W_\rho^{1,p}([-l, l] \times S^1)} \\ \|\xi\|_{W_\rho^{1,p}([-l, l] \times S^1)} &= \|\xi_0\|_{W^{1,p}([-l, l])} + \|\tilde{\xi}\|_{W_\rho^{1,p}([-l, l] \times S^1)}. \end{aligned}$$

For $\eta \in L^p(u^*TM \otimes_J \Lambda^{0,1}([-l, l] \times S^1))$, the norm

$$\|\eta\|_{L_\rho^p([-l, l] \times S^1)} = \|\eta_0\|_{L^p([-l, l])} + \|\tilde{\eta}\|_{L_\rho^p([-l, l] \times S^1)}$$

is defined similarly by using parallel transport to $cm(u)$ and decomposition $\eta = \eta_0 + \tilde{\eta}$. Let

$$L_\rho^p(u^*TM \otimes_J \Lambda^{0,1}([-l, l] \times S^1)) = \left\{ \eta \mid \eta \in L^p([-l, l] \times S^1), \|\eta\|_{L_\rho^p([-l, l] \times S^1)} < \infty \right\}.$$

We define the Banach bundle

$$\mathcal{L}_\chi = \bigcup_{u \in \mathcal{B}_\chi} L_\rho^p(u^*TM \otimes_J \Lambda^{0,1}([-l, l] \times S^1)).$$

Then the Floer operator

$$\bar{\partial}_{J_0, f} : u \mapsto (\partial_\tau u + J_0 \partial_t u + A(\tau)u) \otimes (d\tau - it)$$

gives a Fredholm section \mathcal{F} of the Banach bundle $\mathcal{L}_\chi \rightarrow \mathcal{B}_\chi$.

5.2. L^2 estimate of the right inverse. For any section $\xi \in W^{1,2}(u_\chi^*TM)$, we take the Fourier expansion

$$\xi(\tau, t) = \sum_{k=-\infty}^{\infty} a_k(\tau) e^{2\pi i k t},$$

where $a_k(\tau)$ are vectors in $T_{\chi(\tau)}M$. It is easy to see

$$\xi_0(\tau) = a_0(\tau), \quad \tilde{\xi}(\tau) = \sum_{k \neq 0} a_k(\tau) e^{2\pi i k t}.$$

Let V_0 and \tilde{V} be the L^2 -completions of the spans of zero Fourier mode and of higher Fourier modes respectively, then we have the L^2 -decomposition

$$W^{1,2}(u_\chi^*TM) = V_0 \oplus \tilde{V}$$

where we still denote by V_0, \tilde{V} the intersections

$$V_0 \cap W^{1,2}(u_\chi^*TM), \quad \tilde{V} \cap W^{1,2}(u_\chi^*TM)$$

respectively. We observe that V_0 and \tilde{V} are invariant subspaces of operator $D = \frac{\partial}{\partial \tau} + J_0 \frac{\partial}{\partial t} + A(\tau)$ and so D splits into

$$D = D_0 \oplus \tilde{D} : V_0 \oplus \tilde{V} \rightarrow V_0 \oplus \tilde{V},$$

where $\tilde{D} = D|_{\tilde{V}} : \tilde{V} \rightarrow \tilde{V}$, and $D_0 = D|_{V_0} : V_0 \rightarrow V_0$. Notice that $D_0 = \frac{\partial}{\partial \tau} + A(\tau)$ is exactly the linearized gradient operator L of f .

For the construction of the right inverse Q of D , we use Fourier expansions of η . For any given η , we write

$$\eta(\tau, t) = \sum_{-\infty}^{\infty} b_k(\tau) e^{i2\pi k t}.$$

Now the equation $D\xi = \eta$, i.e. $\partial_\tau \xi + J_0 \partial_t \xi + A(\tau)\xi = \eta$ splits into

$$a'_k(\tau) + (A(\tau) - 2\pi k) a_k(\tau) = b_k(\tau) \quad \text{for all } k \in \mathbb{Z} \quad (5.7)$$

Especially when $k = 0$, it becomes

$$a'_0(\tau) + A(\tau)a_0(\tau) = b_0(\tau). \quad (5.8)$$

We note (5.8) is exactly the linearized gradient flow equation. We can always solve (5.7)

$$a_k(\tau) = e^{-\int_0^\tau (A(\mu) - 2\pi k) d\mu} \left[\int_0^\tau b_k(s) e^{\int_0^s (A(\mu) - 2\pi k) d\mu} ds + C_k \right] \quad (5.9)$$

by the variation of constants with C_k arbitrary constant. Any choice of C_k will produce a right inverse of D .

However for the resulting right inverse to carry uniform bound independent of $\varepsilon > 0$, we need to impose a good boundary condition on a_k , which in turn requires us to make a good choice of the free constants C_k .

For $k = 0$, we put the boundary condition

$$a_0(\pm l) = \xi_0(\pm l) \in d(ev_\pm) (T_{(u_\pm, o_\pm)} \mathcal{M}_2(K_\pm, J_\pm; A_\pm)), \quad (5.10)$$

where $ev_\pm : \mathcal{M}_2(K_\pm, J_\pm; A_\pm) \rightarrow M$ are the evaluation maps, $ev_\pm(u_\pm, o_\pm) = u_\pm(o_\pm)$. The disc-flow-disc transversality condition, Proposition 6.4, enables us to solve this two point boundary problem (5.10).

For $k \neq 0$, we impose one point boundary condition

$$a_k(l) = 0 \text{ if } k > 0; \quad a_k(-l) = 0 \text{ if } k < 0. \quad (5.11)$$

The boundary condition can be always satisfied since the equation (5.8) is a first order linear ODE or one can choose C_k arbitrarily. In fact this one point condition uniquely determines a_k for $k \neq 0$:

$$a_k(\tau) = \begin{cases} -\int_\tau^l b_k(s) e^{\int_\tau^s (A(\mu) - 2\pi k) d\mu} ds & \text{if } k > 0, \\ \int_{-l}^\tau b_k(s) e^{\int_\tau^s (A(\mu) - 2\pi k) d\mu} ds & \text{if } k < 0. \end{cases} \quad (5.12)$$

This constructs the right inverse Q of D and we denote it by $Q(\eta) =: \xi$. From above we see $b_0(\tau) = 0$ if and only if $a_0(\tau) = 0$, thus Q also splits into

$$Q = Q_0 \oplus \tilde{Q} : V_0 \oplus \tilde{V} \rightarrow V_0 \oplus \tilde{V} \quad (5.13)$$

where $Q_0 = Q|_{V_0}$ and $\tilde{Q} = Q|_{\tilde{V}}$.

In fact, the above discussion shows that if the image of Q becomes the subspace $W_0^{1,2}(u_\chi^* TM) := \{\xi \in W_0^{1,2}(u_\chi^* TM) \mid a_k(\ell) = 0 \text{ for } k > 0, \quad a_k(-\ell) = 0, \text{ for } k < 0\}$

then restriction of D to the subspace is an isomorphism with its inverse given by Q .

Remark 5.3. The boundary condition (5.11) is geared more for L^2 estimate of Q rather than matching with the J -holomorphic spheres u_{\pm} . We can't put two-point boundary condition for each $a_k (k \neq 0)$, since we have only one free constant C_k in (5.9). There are lots of choices of the right inverse Q , with various operator norm bound, but for uniform L^2 estimate of Q our choice seems to be the most optimal one, which can be seen in the following estimate (5.14).

We estimate $\|\tilde{\xi}\|_2$. For $k \neq 0$,

$$\begin{aligned}
& (2\pi k)^2 \int_{-l}^l |a_k(\tau)|^2 d\tau \\
& \leq \int_{-l}^l (|a'_k|^2 + (2\pi k)^2 |a_k(\tau)|^2) d\tau \\
& = \int_{-l}^l |a'_k - 2\pi k a_k|^2 d\tau + 2\pi k \int_{-l}^l 2a_k \cdot a'_k d\tau \\
& = \int_{-l}^l |b_k(\tau) - A(\tau)a_k(\tau)|^2 d\tau + 2\pi k \int_{-l}^l \frac{d}{d\tau} |a_k(\tau)|^2 d\tau \\
& = \int_{-l}^l |b_k(\tau) - A(\tau)a_k(\tau)|^2 d\tau + 2\pi k (|a_k(l)|^2 - |a_k(-l)|^2) \\
& \leq 2 \left(\int_{-l}^l |b_k|^2 d\tau + \delta^2 \int_{-l}^l |a_k|^2 d\tau \right) + 2\pi k (|a_k(l)|^2 - |a_k(-l)|^2) \quad (5.14)
\end{aligned}$$

provided $|A(\tau)|_{\infty} < \delta$. By the boundary condition (5.11) of a_k , the second summand of the last inequality is never positive, and noting $\delta < 1 - 1/p < 1$, so we get

$$\int_{-l}^l |a_k(\tau)|^2 d\tau \leq \frac{2}{(2\pi k)^2 - 2\delta^2} \int_{-l}^l |b_k(\tau)|^2 d\tau \leq \int_{-l}^l |b_k(\tau)|^2 d\tau \text{ when } k \neq 0,$$

Summing over $k \neq 0$ we get

$$\|\tilde{\xi}\|_{L^2[-l,l]}^2 \leq \|\tilde{\eta}\|_{L^2[-l,l]}^2. \quad (5.15)$$

From (5.11) and (5.14), we also get

$$0 \leq ((2\pi k)^2 - 2\delta^2) \int_{-l}^l |a_k|^2 \leq 2 \int_{-l}^l |b_k|^2 + 2\pi k (0 - |a_k(-l)|^2)$$

for $k > 0$ and similar inequality for $k < 0$. Hence

$$|a_k(-l)| \leq \sqrt{\frac{1}{k\pi}} \|b_k\|_{L^2([-l,l])} \text{ if } k > 0; \quad |a_k(l)| \leq \sqrt{\frac{-1}{k\pi}} \|b_k\|_{L^2([-l,l])} \text{ if } k < 0.$$

Squaring and summing over $k \neq 0$ we also have

$$\sum_{k>0} k a_k^2(-l) \leq \frac{1}{\pi} \|\tilde{\eta}\|_2^2, \quad \sum_{k<0} |k| a_k^2(l) \leq \frac{1}{\pi} \|\tilde{\eta}\|_2^2. \quad (5.16)$$

Remark 5.4. It would be nice if we can get C^0 estimate $\tilde{\xi}(\pm l, t)$ in terms of $\|\tilde{\eta}\|_2$, but it seems that we can at most get the $W^{\frac{1}{2}, 2}$ norm estimate of $\tilde{\xi}(\pm l, t)$ by the above summation inequalities. However, we will later derive the C^0 estimate of $\tilde{\xi}$ by $W^{1,p}$ estimate and Sobolev embedding.

Remark 5.5. If we extend η to be 0 outside $[-l, l]$, then we can think η is on the full gradient trajectory χ , corresponding to the case where $l = \infty$. Then we can still use the above method to construct $\tilde{\xi}$ with slightly different boundary condition

$$a_k(\infty) = 0 \text{ if } k > 0; \quad a_k(-\infty) = 0 \text{ if } k < 0.$$

(Here $a_k(\pm\infty)$ makes sense, since in their defining integrals $b_k(\tau)$ is compactly supported). For such $\tilde{\xi}$ we gain stronger inequality

$$\|\tilde{\xi}\|_{L^2(-\infty, \infty)}^2 \leq \|\tilde{\eta}\|_{L^2[-l, l]}^2.$$

We choose this $\tilde{\xi}$ in the remaining part of our paper, because the above stronger inequality is needed to construct the approximate right inverse.

Last we estimate $|\xi_0|_{W^{1,p}}$. We notice ξ_0 satisfies the linearized gradient flow equation

$$\frac{\partial}{\partial \tau} \xi_0(\tau) + A(\tau) \xi_0(\tau) = \eta_0(\tau).$$

Note the gradient flow χ can always be extended to a full gradient flow connecting two critical points, whose Fredholm theory and transversality has been well established. Therefore by extending η to be 0 outside $[-l, l]$ and using the right inverse for L_χ on the full gradient trajectory χ , ξ_0 is always solvable and

$$\|\xi_0\|_{L(-\infty, \infty)} \leq C_p \|\eta_0\|_{L^p[-l, l]},$$

especially

$$\|\xi_0\|_{W^{1,p}[-l, l]} \leq C_p \|\eta_0\|_{L^p[-l, l]} \quad (5.17)$$

for some constant C_p ($p \geq 2$).

5.3. L^p estimate of the right inverse. Now assume $p \geq 2$. Without loss of generality we assume l is an integer greater than 1 (The general case $l \geq l_0 > 0$ is similar). Let $B_m = [m - 1/2, m + 1/2] \times S^1$ and $B_m^+ = [m - 1, m + 1] \times S^1$. Given η on $[-l, l]$, we extend it on $[-l - 1, l + 1]$ by letting $\eta = 0$ outside $[-l, l]$. Using the method in the above section we construct $Q(\eta)$, then restrict it on $[-l, l]$.

By the elliptic regularity we can improve L^2 estimate to L^p estimate, if we enlarge the region a bit. In our case the inequality is of the form

$$\|\tilde{\xi}\|_{W^{1,p}(B_m)} \leq C(\|D\tilde{\xi}\|_{L^p(B_m^+)} + \|\tilde{\xi}\|_{L^2(B_m^+)}) \quad (5.18)$$

where the constant C is independent on m . Summing m from $-l$ to l , we get

$$\begin{aligned} \|\tilde{\xi}\|_{W^{1,p}([-l, l] \times S^1)} &\leq C(\|D\tilde{\xi}\|_{L^p([-l-1, l+1] \times S^1)} + \|\tilde{\xi}\|_{L^2([-l-1, l+1] \times S^1)}) \\ &\leq C(\|\tilde{\eta}\|_{L^p([-l, l] \times S^1)} + \|\tilde{\eta}\|_{L^p([-l, l] \times S^1)}) \\ &= 2C\|\tilde{\eta}\|_{L^p([-l, l] \times S^1)} \end{aligned} \quad (5.19)$$

where the second inequality holds because of $\tilde{\eta} = 0$ on $\pm[l, l+1]$ and the L^2 estimate (5.15). Here C is independent of l .

For ξ_0 , we have (5.17). Altogether we get the right inverse

$$Q : L^p([-l, l] \times S^1) \rightarrow W^{1,p}([-l, l] \times S^1); \quad \eta \rightarrow \xi$$

with uniform bound

$$\|Q\eta\|_{W^{1,p}([-l,l] \times S^1)} \leq C \|\eta\|_{L^p([-l,l] \times S^1)}$$

with C independent of the size l .

Remark 5.6. Since the construction of the higher mode of ξ is geared for the uniform right inverse bound, not for the matching condition, we can obtain it by an easier method: We first extend η outside $[-l, l]$ trivially by letting $\eta = 0$ there, then we can apply the method in chapter 3 of [Don] to construct the right inverse of higher mode part of $\frac{\partial}{\partial \tau} + J_0 \frac{\partial}{\partial t} : C^\infty(\mathbb{R} \times S^1, \mathbb{C}^n) \rightarrow C^\infty(\mathbb{R} \times S^1, \mathbb{C}^n)$, since $J_0 \frac{\partial}{\partial t} : C^\infty(S^1, \mathbb{C}^n) \rightarrow C^\infty(S^1, \mathbb{C}^n)$ is τ -independent operator and has nonzero spectrum when restricted on higher mode subspace. Then we simply restrict the obtained $\xi = Q\eta$ on $[-l, l]$. Then the L^2 and L^p estimate for $\|Q\|$ has already been obtained in [Don]. For the operator $D^\varepsilon = \frac{\partial}{\partial \tau} + J_0 \frac{\partial}{\partial t} + A_\varepsilon(\tau)$, it is a ε -small perturbation of $\frac{\partial}{\partial \tau} + J_0 \frac{\partial}{\partial t}$, hence its right inverse bound is inherited from Q . In the next section the right inverse bound can be obtained in the same way, since for small δ , the $(1 + |\tau|)^\delta$ weight amounts to a δ -perturbation of $\frac{\partial}{\partial \tau} + J_0 \frac{\partial}{\partial t}$.

5.4. L_ρ^p estimate of the right inverse. On a fixed domain $[-l, l] \times S^1$, the $W^{1,p}$ norm and the $W_\rho^{1,p}$ norm are equivalent, they defines the same function space. But when $l \rightarrow \infty$, the weighted norm gives better control of the ‘‘Morse-Bott’’ variation. This is a soft technique to get around the point estimate of $\tilde{\xi}(\pm l/\varepsilon, t)$ that we are lacking.

Choose $0 < \delta < 1$. By conjugating with the multiplication of $\rho(\tau)^{\frac{1}{p}}$ where $\rho(\tau) = (1 + |\tau|)^\delta$ is the weighting function, the operator $D : W_\rho^{1,p} \rightarrow L_\rho^p$ is equivalent to $D_\rho : W^{1,p} \rightarrow L^p$, with

$$D_\rho = D - \frac{\left(\rho(\tau)^{\frac{1}{p}}\right)'}{\left(\rho(\tau)\right)^{\frac{1}{p}}} = D - \frac{\delta/p}{1 + |\tau|}.$$

See the following diagram

$$\begin{array}{ccc} W^{1,p} & \xrightarrow{D_\rho} & L^p \\ \rho(\tau)^{\frac{1}{p}} \uparrow & & \uparrow \rho(\tau)^{\frac{1}{p}} \\ W_\rho^{1,p} & \xrightarrow{D} & L_\rho^p \end{array}$$

For the restriction $D = \frac{\partial}{\partial \tau} + J_0 \frac{\partial}{\partial t} + A(\tau)$ to the higher modes \tilde{V} , $D - \frac{\delta/p}{1 + |\tau|}$ is also invertible since the restriction $J_0 \partial_t + A(\tau)$ on

$$\bigoplus_{k \neq 0} E_k \subset L^2(S^1, \mathbb{R}^n)$$

has its spectrum outside $(-1, 1) \subset \mathbb{R}$ and we have choose that $0 < \frac{\delta}{p} < 1$. Similarly to (5.19), we obtain

$$\|\tilde{\xi}\|_{W_\rho^{1,p}([-l,l] \times S^1)} \leq 2C \|\tilde{\eta}\|_{L_\rho^p([-l,l] \times S^1)} \quad (5.20)$$

where C is independent on l .

The gain of this $W_\rho^{1,p}$ -estimate of $\tilde{\xi}$ is the following pointwise decay estimate:

$$|\tilde{\xi}(\tau, t)|_{C^0} \leq \frac{1}{|\tau|^\delta} \|\tilde{\xi}\|_{W_\rho^{1,p}([-l, l] \times S^1)} \leq \frac{2C}{|\tau|^\delta} \|\tilde{\eta}\|_{L_\rho^p([-l, l] \times S^1)} \quad (5.21)$$

through by Sobolev embedding $W^{1,p} \hookrightarrow C^0$

For the zero-mode, by Sobolev embedding $W^{1,p}([-l, l]) \hookrightarrow C^\gamma([-l, l])$ ($l \geq l_0 > 0$) with $\gamma = 1 - \frac{1}{p}$, we have

$$\begin{aligned} |\xi_0(\tau) - \xi_0(\pm l)|_{C^0} &\leq C |\tau \pm l|^\gamma \|\xi_0\|_{W^{1,p}([-l, l])} \\ &\leq 2C |\tau \pm l|^\delta \|\eta_0\|_{L^p([-l, l])} \end{aligned} \quad (5.22)$$

when $|\tau \pm l| < 1$ and $0 < \delta < 1 - 1/p$.

Remark 5.7. The power weight $\rho(\tau) = (1 + |\tau|)^\delta$ takes care of the decay of the high modes. We choose this weight because the *gradient segment* χ converges to its noncritical endpoints p_\pm in linear order, not in exponential order. If the χ is a *full gradient trajectory* connecting two nondegenerate Morse critical points, we do not need any weight because the higher mode $\tilde{\xi}$ with finite $W^{1,p}$ norm automatically vanishes at $\tau = \pm\infty$. If one of the critical end points of χ is Morse-Bott, we need to put the exponential weight to capture the correct convergence rate. The power weight and exponential weight can be unified by one function, say $\left(\int_0^{|\tau|} \left|\frac{1}{\nabla f(\chi)}\right| d\tau\right)^\delta + 1$.

5.5. ε -reparametrization of the gradient segment and adiabatic weight. When $\varepsilon \rightarrow 0$, we reparametrize the gradient segment $\chi(\tau)$ to $\chi(\varepsilon\tau)$ for gluing. Along $\chi(\varepsilon\tau)$ we define ε -family of Banach manifolds as following:

$$\|\xi\|_{W_{\rho_\varepsilon}^{1,p}([-l/\varepsilon, l/\varepsilon] \times S^1)} = \|\xi_0(\tau)\|_{W_\varepsilon^{1,p}([-l/\varepsilon, l/\varepsilon])} + \|\tilde{\xi}(\tau)\|_{W_{\rho_\varepsilon}^{1,p}([-l/\varepsilon, l/\varepsilon] \times S^1)},$$

and

$$\|\eta\|_{L_{\rho_\varepsilon}^p([-l/\varepsilon, l/\varepsilon] \times S^1)} = \|\eta_0(\tau)\|_{L_\varepsilon^p([-l/\varepsilon, l/\varepsilon])} + \|\tilde{\eta}(\tau)\|_{L_{\rho_\varepsilon}^p([-l/\varepsilon, l/\varepsilon] \times S^1)}.$$

Here for ξ_0 and η_0 , the so called geometric ε -weighted norm $W_\varepsilon^{1,p}$ and L_ε^p are

$$\|\xi_0\|_{W_\varepsilon^{1,p}[-l/\varepsilon, l/\varepsilon]}^p = \int_{-l/\varepsilon}^{l/\varepsilon} (\varepsilon |\xi_0|^p + \varepsilon^{1-p} |\nabla \xi_0|^p) d\tau$$

and

$$\|\eta_0\|_{L_\varepsilon^p[-l/\varepsilon, l/\varepsilon]}^p = \int_{-l/\varepsilon}^{l/\varepsilon} \varepsilon^{1-p} |\eta_0|^p d\tau.$$

The geometric ε -weight is useful in the adiabatic limit problems, for example in [FO1]. One loses uniform right inverse bound of $\frac{\partial}{\partial \tau} + \varepsilon \nabla \text{grad} f$ on χ_ε if just use usual $W^{1,p}$ norm, since the spectrum of $\varepsilon \nabla \text{grad} f$ goes to 0 as $\varepsilon \rightarrow 0$. The norm $\|\cdot\|_{W_\varepsilon^{1,p}[-l/\varepsilon, l/\varepsilon]}$ is just the usual Sobolev norm after the reparameterization

$$R_\varepsilon : \chi_\varepsilon \rightarrow \chi, \chi_\varepsilon(\tau') \rightarrow \chi(\tau), \text{ where } \tau' = \varepsilon\tau, \tau' \in [-l/\varepsilon, l/\varepsilon].$$

Indeed, letting

$$\widehat{\xi}_0(\tau) = (R_\varepsilon)^* \xi_0 = \xi_0(\varepsilon\tau),$$

then one can verify

$$\|\xi_0\|_{W_\varepsilon^{1,p}[-l/\varepsilon, l/\varepsilon]} = \|\widehat{\xi}_0\|_{W^{1,p}[-l, l]}.$$

In other words, under this norm

$$R_\varepsilon : W_\varepsilon^{1,p}([-l/\varepsilon, l/\varepsilon], M) \rightarrow W^{1,p}([-l, l], M)$$

is an isometry between two Banach manifolds. Driven by the change of L^p -norm

$$\begin{aligned} & \int_{-l}^l \left| \frac{\partial \chi}{\partial \tau}(\tau) + \text{grad} f(\chi(\tau)) \right|^p d\tau \\ &= \int_{-l/\varepsilon}^{l/\varepsilon} \left| \frac{1}{\varepsilon} \frac{\partial \chi_\varepsilon}{\partial \tau'}(\tau') + \text{grad} f(\chi_\varepsilon(\tau')) \right|^p \varepsilon d\tau' \\ &= \int_{-l/\varepsilon}^{l/\varepsilon} \left| \frac{\partial \chi_\varepsilon}{\partial \tau'}(\tau') + \varepsilon \text{grad} f(\chi_\varepsilon(\tau')) \right|^p \varepsilon^{1-p} d\tau', \end{aligned}$$

under the reparameterization, we define

$$\|\eta_0\|_{L_\varepsilon^p[-l/\varepsilon, l/\varepsilon]}^p = \int_{-l/\varepsilon}^{l/\varepsilon} \varepsilon^{1-p} |\eta_0|^p d\tau,$$

because then the two sections

$$\begin{aligned} \widehat{\eta}_0 &= \frac{\partial}{\partial \tau} + \text{grad} f : W^{1,p}([-l, l], M) \rightarrow \bigcup_{\chi \in W^{1,p}([-l, l], M)} L^p([-l, l], \chi^* TM) \\ \eta_0 &= \frac{\partial}{\partial \tau'} + \varepsilon \text{grad} f : W_\varepsilon^{1,p}([-l/\varepsilon, l/\varepsilon], M) \rightarrow \bigcup_{\chi_\varepsilon \in W_\varepsilon^{1,p}([-l/\varepsilon, l/\varepsilon], M)} L_\varepsilon^p([-l/\varepsilon, l/\varepsilon], \chi_\varepsilon^* M) \end{aligned}$$

in the two Banach bundles are isometrically conjugate to each other. The relation between the two sections is

$$\eta_0(\chi_\varepsilon(\tau')) = \varepsilon \widehat{\eta}_0(\chi(\tau)). \quad (5.23)$$

It is easy to check that under these norms, the right inverse Q_0^ε of the linearized gradient operator

$$D_0^\varepsilon = \frac{\partial}{\partial \tau} + \varepsilon \nabla \text{grad} f : W_\varepsilon^{1,p} \rightarrow L_\varepsilon^p$$

has uniform operator bounds over $\varepsilon > 0$, since we have

$$\|Q_0^\varepsilon\| = \|Q_0\| \quad (5.24)$$

for all ε . The Sobolev constant

$$c_p^\varepsilon := \sup_{\xi_0 \neq 0} \frac{|\xi_0|}{\|\xi_0\|_{W_\varepsilon^{1,p}([-l/\varepsilon, l/\varepsilon])}}} = \sup_{\xi_0 \neq 0} \frac{|\widehat{\xi}_0|}{\|\widehat{\xi}_0\|_{W^{1,p}([-l, l])}}$$

is also uniform for all ε , since it is equal to the $W^{1,p}$ Sobolev constant of $\|\widehat{\xi}_0\|$ on $[-l, l]$, and we have assumed $l \geq l_0 > 0$.

The power weight $\rho_\varepsilon(\tau)$ here is transformed to

$$\rho_\varepsilon(\tau) = \varepsilon^{1-p} \rho(\tau) = \varepsilon^{1-p} (1 + |\tau|)^\delta,$$

and it is used to define

$$\begin{aligned} \|\tilde{\xi}(\tau)\|_{W_{\rho_\varepsilon}^{1,p}([-l/\varepsilon, l/\varepsilon] \times S^1)} &= \|\tilde{\xi}(\tau) (\rho_\varepsilon(\tau))^{-\frac{1}{p}}\|_{W^{1,p}([-l/\varepsilon, l/\varepsilon] \times S^1)}, \\ \|\tilde{\eta}(\tau)\|_{L_{\rho_\varepsilon}^p([-l/\varepsilon, l/\varepsilon] \times S^1)} &= \|\tilde{\eta}(\tau) (\rho_\varepsilon(\tau))^{-\frac{1}{p}}\|_{L^p([-l/\varepsilon, l/\varepsilon] \times S^1)}. \end{aligned}$$

The ε^{1-p} factor in $\rho_\varepsilon(\tau)$ is to make the norms $W_{\rho_\varepsilon}^{1,p}$ and $W_\varepsilon^{1,p}$ comparable, which is important later in our right inverse estimate via the weight comparison. By conjugation with the multiplication operator by the weight function $\rho_\varepsilon(\tau)$,

$$D^\varepsilon = \frac{\partial}{\partial \tau} + J_0 \frac{\partial}{\partial t} + \varepsilon \nabla f(\chi'_\varepsilon) : W^{1,p}([-l/\varepsilon, l/\varepsilon] \times S^1) \rightarrow L^p([-l/\varepsilon, l/\varepsilon] \times S^1)$$

is equivalent to $D_{\rho_\varepsilon} : W_{\rho_\varepsilon}^{1,p} \rightarrow L_{\rho_\varepsilon}^p$, with

$$D_{\rho_\varepsilon} = D_\varepsilon - \frac{(\rho_\varepsilon(\tau)^{\frac{1}{p}})'}{\rho_\varepsilon(\tau)^{\frac{1}{p}}} = D_\varepsilon - \frac{\delta/p}{1+|\tau|}.$$

Since

$$\frac{\|\tilde{\xi}(\tau)\|_{W_{\rho_\varepsilon}^{1,p}([-l/\varepsilon, l/\varepsilon] \times S^1)}}{\|\tilde{\eta}(\tau)\|_{L_{\rho_\varepsilon}^p([-l/\varepsilon, l/\varepsilon] \times S^1)}} = \frac{\|\tilde{\xi}(\tau)\|_{W_\varepsilon^{1,p}([-l/\varepsilon, l/\varepsilon] \times S^1)}}{\|\tilde{\eta}(\tau)\|_{L_\varepsilon^p([-l/\varepsilon, l/\varepsilon] \times S^1)}}$$

the right inverse bound obtained in $W_\varepsilon^{1,p}$ setting passes to the $W_{\rho_\varepsilon}^{1,p}$ setting.

We remark the following useful inequality for ξ_0 on $\chi(\varepsilon\tau)$, by the Sobolev embedding $W^{1,p}([0, l]) \hookrightarrow C^\gamma([0, l])$ with $\gamma = 1 - \frac{1}{p}$:

$$\begin{aligned} |\xi_0(\tau) - \xi_0(\pm l/\varepsilon)| &= |\widehat{\xi}_0(\varepsilon\tau) - \widehat{\xi}_0(\pm l)| \\ &\leq C(l) |\varepsilon\tau - \pm l|^\gamma \|\widehat{\xi}_0\|_{W^{1,p}[0, l]} \\ &= C(l) |\varepsilon\tau \pm l|^\gamma \|\xi_0\|_{W_\varepsilon^{1,p}[-l/\varepsilon, l/\varepsilon]} \end{aligned} \quad (5.25)$$

Especially for $\tau = \pm(l/\varepsilon - T(\varepsilon))$, where $T(\varepsilon) = \frac{1}{3} \frac{p-1}{\delta} S(\varepsilon)$, we have

$$\begin{aligned} |\xi_0(\tau) - \xi_0(\pm l/\varepsilon)| &\leq C(l) (\varepsilon T(\varepsilon))^\gamma \|\xi_0\|_{W_\varepsilon^{1,p}[-l/\varepsilon, l/\varepsilon]} \\ &\leq C(l) |\varepsilon \ln \varepsilon|^\gamma \|\xi_0\|_{W_\varepsilon^{1,p}[-l/\varepsilon, l/\varepsilon]} \\ &\leq C(l) \varepsilon^{\tilde{\gamma}} \|\xi_0\|_{W_\varepsilon^{1,p}[-l/\varepsilon, l/\varepsilon]} \end{aligned}$$

for any $0 < \tilde{\gamma} < 1 - \frac{1}{p}$, if ε is sufficiently small.

6. MODULI SPACE OF “DISC-FLOW-DISC” CONFIGURATIONS

This subsection is the first stage of the deformation of the parameterized moduli space entering in the construction of the chain homotopy map between $\Psi \circ \Phi$ and the identity on $HF(H, J)$ in [PSS]. The material in the first half of this section is largely taken from section 5.1 [OZ1] with slight modifications.

A “disk-flow-disk” configuration consists of two perturbed J -holomorphic discs joined by a gradient flow line between their marked points. In this section we will define the moduli space of such configurations.

For notation brevity, we just denote

$$\mathcal{M}^l(K^\pm, J^\pm; [z_-, w_-]; f; [z_+, w_+]; A_\pm) = \mathcal{M}^l([z_-, w_-]; f; [z_+, w_+]; A_\pm),$$

omitting the Floer datum (K^\pm, J^\pm) , as long as it does not cause confusion.

Given the two moduli spaces $\mathcal{M}([z_-, w_-]; A_-)$ and $\mathcal{M}([z_+, w_+]; A_+)$ and the Morse function f , we define the moduli space

$$\mathcal{M}^l([z_-, w_-]; f; [z_+, w_+]; A_\pm)$$

by the set consisting of “disk-flow-disk” configurations (u_-, χ, u_+) of *flow time* $2l$ such that

$$\mathcal{M}^l([z_-, w_-]; f; [z_+, w_+]; A_\pm) = \left\{ (u_-, \chi, u_+) \left| \begin{array}{l} u_\pm \in \mathcal{M}(K^\pm, J^\pm; \bar{z}_\pm; A_\pm), \\ [u_\pm \# w_\pm] = A_\pm, \chi : [-l, l] \rightarrow M, \\ \dot{\chi} - \nabla f(\chi) = 0, \\ u_-(o_-) = \chi(-l), u_+(o_+) = \chi(l). \end{array} \right. \right\} \quad (6.1)$$

Then the moduli space of “disk-flow-disk” configurations is defined to be

$$\mathcal{M}^{para}([z_-, w_-]; f; [z_+, w_+]; A_\pm) := \bigcup_{l \geq l_0} \mathcal{M}^l([z_-, w_-]; f; [z_+, w_+]; A_\pm). \quad (6.2)$$

We now provide the off-shell formulation of the “disk-flow-disk” moduli spaces. We first provide the Banach manifold hosting $\mathcal{M}^l([z_-, w_-]; f; [z_+, w_+]; A_\pm)$. We define

$$\begin{aligned} \mathcal{B}_l^{dfd}(z_-, z_+) &:= \{(u_-, \chi, u_+) \mid u_\pm \in W^{1,p}(\dot{\Sigma}, M; z_\pm), \\ &\quad \chi \in W^{1,p}([-l, l], M), u_-(o_-) = \chi(-l), u_+(o_+) = \chi(l)\} \end{aligned} \quad (6.3)$$

for $p > 2$. For any $u \in \mathcal{B}_l^{dfd}(z_-, z_+)$, the tangent space is

$$T_u \mathcal{B}_l^{dfd}(z_-, z_+) = \left\{ (\xi_-, a, \xi_+) \mid \xi_\pm \in W^{1,p}(u_\pm^* TM), \xi_-(o_-) = a(-l), \xi_+(o_+) = a(l) \right\}.$$

Then for each $u = (u_-, \chi, u_+) \in \mathcal{B}_l^{dfd}(z_-, z_+)$, we define

$$L_u^p(z_-, z_+) = L^p(\Lambda^{(0,1)} u^* TM)$$

and form the Banach bundle

$$\mathcal{L}_l^{dfd} = \bigcup_{u \in \mathcal{B}_l^{dfd}(z_-, z_+)} L_u^p(z_-, z_+)$$

over $\mathcal{B}_l^{dfd}(z_-, z_+)$. Here the superscript ‘dfd’ stands for ‘disk-flow-disk’. We refer to [Fl1] for a more detailed description of the asymptotic behavior of the elements in $\mathcal{B}_l^{dfd}(z_-, z_+)$ in the context of Floer moduli spaces.

For $u = (u_-, \chi, u_+) \in \mathcal{B}_l^{dfd}(z_-, z_+)$, its tangent space $T_u \mathcal{B}_l^{dfd}$ consists of $\xi = (\xi_-, a, \xi_+)$, where $\xi_\pm \in W^{1,p}(u_\pm^* TM)$, $a \in W^{1,p}(\chi^* TM)$, with the matching condition

$$\xi_-(o_-) = a(-l), \quad \xi_+(o_+) = a(l) \quad (6.4)$$

We denote the set of such ξ as $W_u^{1,p}(z_-, z_+)$.

We let

$$\mathcal{B}^{dfd}(z_-, z_+) = \bigcup_{l \geq l_0} \mathcal{B}_l^{dfd}(z_-, z_+) \quad \text{and} \quad \mathcal{L}^{dfd}(z_-, z_+) = \bigcup_{l \geq l_0} \mathcal{L}_l^{dfd}(z_-, z_+)$$

Remark 6.1. If we regard u in $\mathcal{B}^{dfd}(z_-, z_+)$ instead of $\mathcal{B}_l^{dfd}(z_-, z_+)$, then

$$T_u \mathcal{B}^{dfd}(z_-, z_+) \simeq T_u \mathcal{B}_l^{dfd}(z_-, z_+) \times T_l \mathbb{R},$$

namely the tangent space of u in $\mathcal{B}^{dfd}(z_-, z_+)$ consists of $\xi = (\xi_-, a, \xi_+, \mu)$, where $\xi_\pm \in W^{1,p}(u_\pm^* TM)$, $a \in W^{1,p}(\chi^* TM)$, and $\mu \in T_l \mathbb{R} \cong \mathbb{R}$, with the matching condition

$$\xi_-(o_-) = a(-l) - \frac{\mu}{l} \dot{\chi}(-l), \quad \xi_+(o_+) = a(l) + \frac{\mu}{l} \dot{\chi}(l). \quad (6.5)$$

Here the μ comes from the variation of the length l of the domain of gradient flows. For $\mu \in T_l \mathbb{R}$, the induced path in $\mathcal{B}^{dfd}(z_-, z_+)$, starting from $u = (u_-, \chi, u_+) \in \mathcal{B}_l^{dfd}(z_-, z_+)$, is $u_s = (u_-, \chi_s, u_+) \in \mathcal{B}_{(l-s\mu)}^{dfd}(z_-, z_+)$, where $\chi_s(\tau')$ is the reparameterization of $\chi(\tau)$,

$$\chi_s(\tau') := \chi\left(\frac{l\tau'}{l-s\mu}\right), \quad \tau' \in [-(l-s\mu), l-s\mu],$$

for s nearby 0. There is a canonical way to associate points on χ to points on χ_s :

$$\chi(\tau) \longleftrightarrow \chi_s(\tau'), \quad \text{where } \tau = \frac{l\tau'}{l-s\mu}.$$

They all correspond to the same point on the image of χ .

Now we fix $l > 0$ and consider a natural section

$$e : \mathcal{B}_l^{dfd}(z_-, z_+) \rightarrow \mathcal{L}_l^{dfd}(z_-, z_+) \quad (6.6)$$

such that $e(u) \in L_u^p(z_-, z_+)$ is given by

$$e(u) = (\bar{\partial}_{(K_-, J_-)} u_-, \dot{\chi} - \nabla f(\chi), \bar{\partial}_{(K_+, J_+)} u_+)$$

where the u_{\pm} and χ satisfy the matching condition in (6.3). The linearization of e at $u \in e^{-1}(0) = \mathcal{M}^l([z_-, w_-]; f; [z_+, w_+]; A_{\pm})$ induces a linear operator

$$E(u) := D_u e : T_u \mathcal{B}_l^{dfd}(z_-, z_+) \rightarrow L_u^p(z_-, z_+) \quad (6.7)$$

where we have

$$T_u \mathcal{B}_l^{dfd}(z_-, z_+) = \left\{ (\xi_-, a, \xi_+) \mid \xi_{\pm} \in W^{1,p}(u_{\pm}^* TM), \xi_-(o_-) = a(-l), \xi_+(o_+) = a(l) \right\}$$

and the value $D_u e(\xi) =: \eta$ has the expression

$$\eta = (\eta_-, b, \eta_+) = \left(D_{u_-} \bar{\partial}_{(K_-, J_-)}(\xi_-), \frac{Da}{d\tau} - \nabla_a \text{grad}(f), D_{u_+} \bar{\partial}_{(K_+, J_+)}(\xi_+) \right)$$

for $\xi = (\xi_-, a, \xi_+)$. We remark that if we regard $u \in \mathcal{B}^{dfd}(z_-, z_+)$, then for $(\xi_-, a, \xi_+, \mu) \in T_u \mathcal{B}^{dfd}(z_-, z_+)$, $D_u e(\xi) =: \eta$ has the expression $\eta = (\eta_-, b, \eta_+)$ where

$$b = \frac{D}{d\tau} a - \nabla_a \text{grad}(f) + \frac{\mu}{l} \dot{\chi}(\tau). \quad (6.8)$$

Here we have used that

$$\begin{aligned} D_u e(0, 0, 0, \mu) &= \left. \frac{d}{ds} \right|_{s=0} \left[\frac{d\chi_s}{d\tau'}(\tau') - \nabla f(\chi_s(\tau')) \right] \\ &= \left. \frac{d}{ds} \right|_{s=0} \left[\frac{d\chi}{d\tau'}(\tau) - \nabla f(\chi(\tau)) \right] \\ &= \left. \frac{d}{ds} \right|_{s=0} \left[\frac{l}{l-s\mu} \frac{d\chi}{d\tau}(\tau) \right] = \frac{\mu}{l} \dot{\chi}(\tau). \end{aligned}$$

For the simplicity of notation, we denote

$$\begin{aligned} W_u^{1,p}(z_-, z_+; dfd) &:= T_u \mathcal{B}_l^{dfd}(z_-, z_+) \\ &\subset W^{1,p}(u_-^* TM) \times W^{1,p}(\chi^* TM) \times W^{1,p}(u_+^* TM). \end{aligned}$$

Now we show $E(u)$ is Fredholm and compute its index.

Proposition 6.2. *The operator $E(u)$ is a Fredholm operator and we have*

$$\text{Index } E(u) = \mu_{H_-}([z_-, w_-]) - \mu_{H_+}([z_+, w_+]) + 2c_1(A_-) + 2c_1(A_+). \quad (6.9)$$

for any

$$u = (u_-, \chi, u_+) \in \mathcal{M}^l([z_-, w_-]; f; [z_+, w_+]; A_{\pm}).$$

Proof. We compute the kernel and the cokernel of

$$E(u) : W_u^{1,p}(z_-, z_+; dfd) \rightarrow L_u^p(z_-, z_+).$$

By the matching condition (6.5) it is clear that

$$\begin{aligned} \ker E(u) = & \left\{ (\xi_-, \xi_+, a) \mid \xi_{\pm} \in \ker D_{u_{\pm}} \bar{\partial}_{(K^{\pm}, J^{\pm})}, \right. \\ & \left. \frac{Da}{\partial \tau} - \nabla_a \text{grad } f(\chi) = 0, \xi_-(o_-) = a(-l), \xi_+(o_+) = a(l) \right\}, \end{aligned} \quad (6.10)$$

It is easy to see

$$\xi_+(o_+) = d\phi_f^{2l} \cdot \xi_-(o_-)$$

for any $(\xi_-, \xi_+, a) \in \ker E(u)$ noticing that a is determined by its initial value $a(-l)$ and by the equation

$$\frac{Da}{\partial \tau} - \nabla_a \text{grad } f(\chi) = 0. \quad (6.11)$$

Therefore the $(\xi_-, \xi_+, a) \in \ker E(u)$ have 1-1 correspondence to the points of $\left[d \left(\phi_f^{2l} ev_- \times ev_+ \right) \right]^{-1} \left(\Delta_{u_+(o_+)}^{tn} \right)$, where

$$d \left(\phi_f^{2l} ev_- \times ev_+ \right) : \ker D_{u_-} \bar{\partial}_{(K^-, J^-)} \times \ker D_{u_+} \bar{\partial}_{(K_+, J_+)} \rightarrow T_{u_+(o_+)} M \times T_{u_+(o_+)} M$$

and $\Delta_{u_+(o_+)}^{tn} = \{(v, v) \mid v \in T_{u_+(o_+)} M\} \subset T_{u_+(o_+)} M \times T_{u_+(o_+)} M$, which has codimension $2n$. Let the subspaces

$$V_{\pm} := ev_{o_{\pm}} \left(\ker D_{u_{\pm}} \bar{\partial}_{(K^{\pm}, J^{\pm})} \right) \subset T_{u_{\pm}(o_{\pm})} M.$$

We have

$$\begin{aligned} \dim \ker E(u) &= \dim \ker D_{u_-} \bar{\partial}_{(K^-, J^-)} + \dim \ker D_{u_+} \bar{\partial}_{(K_+, J_+)} - 2n \\ &\quad + 4n - \dim \left(\left(d\phi_f^{2l} \cdot V_- \right) \times V_+ + \Delta_{u_+(o_+)}^{tn} \right) \\ &= \dim \ker D_{u_-} \bar{\partial}_{(K^-, J^-)} + \dim \ker D_{u_+} \bar{\partial}_{(K_+, J_+)} \\ &\quad - \dim \left(d\phi_f^{2l} \cdot V_- + V_+ \right) \end{aligned} \quad (6.12)$$

where in the last identity we have used the linear algebra fact that

$$\dim(A \times B + \Delta) = \dim \Delta + \dim(A - B)$$

for linear subspaces A, B in V and diagonal Δ in $V \times V$. Next we compute the cokernel of $E(u)$. Let $E(u)^*$ be the adjoint operator of $E(u)$, such that

$$E(u)^* : L_u^p(z_-, z_+)^* \rightarrow W^{1,p}(z_-, z_+; dfd)^*.$$

Using the nondegenerate L^2 pairing

$$L^p(\Lambda^{(0,1)} u^* TM) \times L^q(\Lambda^{(1,0)} u^* TM) \rightarrow \mathbb{R}$$

we identify $L_u^p(z_-, z_+)^*$ with $L^q(\Lambda^{(1,0)}u^*TM)$. On the other hand, we can identify $W^{1,p}(z_-, z_+; dfd)^*$ with

$$\begin{aligned} & W^{-1,q}(z_-, z_+; dfd) \\ &:= \{(\xi_-, a, \xi_+) \in W^{1,p}(z_-, z_+) \mid \xi_-(o_-) = a(-l), \xi_+(o_+) = a(l)\}^\perp \end{aligned}$$

in the direct product

$$W^{-1,q}(z_-, z_+) = W^{-1,q}(u_-^*TM) \times W^{-1,q}(\chi^*TM) \times W^{-1,q}(u_+^*TM)$$

where $(\cdot)^\perp$ denotes the L^2 -orthogonal complement. Here we have $1 < q < 2$ since $2 < p < \infty$.

We denote by

$$E(u)^\dagger : L^q(\Lambda^{(1,0)}u^*TM) \rightarrow W^{-1,q}(z_-, z_+; dfd)$$

the corresponding L^2 -adjoint with respect to these identifications.

Now we derive the formula for $E(u)^\dagger$. Recall by definition, we have

$$\langle E(u)\xi, \eta \rangle = \langle \xi, E(u)^\dagger \eta \rangle.$$

Then for any given $\eta := (\eta_-, b, \eta_+) \in \ker E^\dagger(u) \subset L^q(z_-, z_+)$ it satisfies

$$\begin{aligned} 0 &= \int_{-l}^l \left\langle \frac{Da}{\partial\tau} - \nabla \text{grad } f(\chi)a, b \right\rangle \\ &\quad + \int_{\Sigma_-} \langle D_{u_-} \bar{\partial}_{(K^-, J^-)} \xi_-, \eta_- \rangle + \int_{\Sigma_+} \langle D_{u_+} \bar{\partial}_{(K^+, J^+)} \xi_+, \eta_+ \rangle \end{aligned}$$

for all $(\xi_+, a, \xi_-) \in W_u^{1,p}(z_-, z_+; dfd)$, i.e., for all the triples satisfying the matching condition

$$\xi_-(o_-) = a(-l), \quad \xi_+(o_+) = a(l). \quad (6.13)$$

Integrating by parts, we have

$$\begin{aligned} 0 &= \langle a(l), b(l) \rangle - \langle a(-l), b(-l) \rangle + \int_{-l}^l \left\langle -\frac{Db}{\partial\tau} - \nabla \text{grad } f(\chi)b, a \right\rangle \\ &\quad - \int_{\Sigma_-} \langle (D_{u_-} \bar{\partial}_{(K^-, J^-)})^\dagger \eta_-, \xi_- \rangle - \int_{\Sigma_+} \langle (D_{u_+} \bar{\partial}_{(K^+, J^+)})^\dagger \eta_+, \xi \rangle. \end{aligned} \quad (6.14)$$

Here $D_{u_\pm} \bar{\partial}_{(K^\pm, J^\pm)}$ is the formal adjoint of $D_{u_\pm} \bar{\partial}_{(K^\pm, J^\pm)}$ which has its symbol of that of the Dolbeault operator $\bar{\partial}$ (near $z = 0$ in \mathbb{C}) and so elliptic. *Here we note that we are using a metric on $\Sigma_\pm \cong \mathbb{C}$ that is standard near the origin o_\pm and cylindrical near the end.*

Substituting (6.13) into this we can rewrite (6.14) into

$$\begin{aligned} 0 &= \int_{-l}^l \left\langle -\frac{Db}{\partial\tau} - \nabla \text{grad } f(\chi)b, a \right\rangle + \langle \xi_+(o_+), b(\varepsilon) \rangle - \langle \xi_-(o_-), b(0) \rangle \\ &\quad - \int_{\Sigma_-} \langle (D_{u_-} \bar{\partial}_{(K^-, J^-)})^\dagger \eta_-, \xi_- \rangle - \int_{\Sigma_+} \langle (D_{u_+} \bar{\partial}_{(K^+, J^+)})^\dagger \eta_+, \xi \rangle. \end{aligned}$$

Note a can be varied arbitrarily on the interior $(-l, l)$ and can be matched to any given $\xi_\pm(o_\pm)$ at $-l, l$. Therefore considering the variation $\xi_- = \xi_+ = 0$, we derive that b must satisfy

$$\left\langle -\frac{Db}{\partial\tau} - \nabla \text{grad } f(\chi)b, a \right\rangle = 0$$

for all a with $a(-l) = 0 = a(l)$. Therefore b satisfies

$$-\frac{Db}{\partial\tau} - \nabla \text{grad } f(\chi)b = 0 \quad (6.15)$$

on $[-l, l]$ first in the distribution sense and then in the classical sense by the bootstrap regularity of the ODE (6.15) and so it is smooth. Let

$$P^\dagger : T_{u_-(o_-)}M \rightarrow T_{u_+(o_+)}M, \quad b(-l) \rightarrow b(l)$$

be the linear map for solutions b of ODE (6.15), then

$$b(l) = P^\dagger b(-l).$$

Substituting (6.15) into the above we obtain

$$\begin{aligned} 0 &= -\langle \xi_-(o_-), b(-l) \rangle + \int_{\dot{\Sigma}_-} \langle (D_{u_-} \bar{\partial}_{(K^-, J^-)})^\dagger \eta_-, \xi_- \rangle \\ &\quad + \langle \xi_+(o_+), b(l) \rangle + \int_{\dot{\Sigma}_+} \langle (D_{u_+} \bar{\partial}_{(K^+, J^+)})^\dagger \eta_+, \xi_+ \rangle. \end{aligned}$$

Now we can vary ξ_\pm independently and hence we have

$$\begin{aligned} 0 &= -\langle \xi_-(o_-), b(-l) \rangle - \int_{\dot{\Sigma}_-} \langle (D_{u_-} \bar{\partial}_{(K^-, J^-)})^\dagger \eta_-, \xi_- \rangle \\ 0 &= \langle \xi_+(o_+), b(l) \rangle + \int_{\dot{\Sigma}_+} \langle (D_{u_+} \bar{\partial}_{(K^+, J^+)})^\dagger \eta_+, \xi_+ \rangle \end{aligned}$$

and hence

$$\begin{aligned} (D_{u_-} \bar{\partial}_{(K^-, J^-)})^\dagger \eta_- - b(-l) \delta_{o_-} &= 0 \\ (D_{u_+} \bar{\partial}_{(K^+, J^+)})^\dagger \eta_+ + b(l) \delta_{o_+} &= 0. \end{aligned} \quad (6.16)$$

Here δ_o denotes the Dirac-delta measure supported at the point $\{o\} \subset \Sigma$. Due to the choice of our metric on the domain \mathbb{C} of u_\pm , η_\pm must have the singularity of the type $\frac{1}{z}$ which is the fundamental solution to $\partial\eta = \bar{b}\delta_o$ which lies in L^q for any $1 < q < 2$. Therefore one can solve the distributional equation

$$(D_{u_\pm} \bar{\partial}_{(K^\pm, J^\pm)})^\dagger \eta = \bar{b} \cdot \delta_{o_\pm}$$

provided that \bar{b} satisfies the Fredholm alternative:

$$\langle \bar{b}, \xi_\pm(o_\pm) \rangle = \int_{\dot{\Sigma}_\pm} \langle \bar{b} \cdot \delta_{o_\pm}, \xi_\pm \rangle = 0$$

for all $\xi_\pm \in \ker((D_{u_\pm} \bar{\partial}_{(K^\pm, J^\pm)})^\dagger)^\dagger = \ker D_{u_\pm} \bar{\partial}_{(K^\pm, J^\pm)}$. Namely $\bar{b} \in (V_\pm)^\perp$, where

$$V_\pm := \text{ev}_{o_\pm}(\ker D_{u_\pm} \bar{\partial}_{(K^\pm, J^\pm)}).$$

We fix such a solution denoted by $\eta_{\bar{b}} \in L^q$.

Then (6.16) can be written as

$$(D_{u_-} \bar{\partial}_{(K^-, J^-)})^\dagger (\eta_- - \eta_{b(-l)}) = 0, \quad (D_{u_+} \bar{\partial}_{(K^+, J^+)})^\dagger (\eta_+ + \eta_{b(l)}) = 0$$

i.e.,

$$\begin{aligned} \eta_- + \eta_{b(-l)} &\in \ker(D_{u_-} \bar{\partial}_{(K^-, J^-)})^\dagger, \\ \eta_+ - \eta_{b(l)} &\in \ker(D_{u_+} \bar{\partial}_{(K^+, J^+)})^\dagger. \end{aligned}$$

Therefore we have the exact sequence

$$\begin{aligned} 0 &\rightarrow \text{Graph } P^\dagger \cap (V_-^\perp \times V_+^\perp) \xrightarrow{i} \ker E^\dagger(u) \\ &\xrightarrow{j} \ker(D_{u_+} \bar{\partial}_{(K_+, J_+)})^\dagger \oplus \ker(D_{u_-} \bar{\partial}_{(K_-, J_-)})^\dagger \rightarrow 0 : \end{aligned}$$

Here the first homomorphism is the map

$$i(b_1, b_2) = (\eta_{b_1}, b_{b_1}, -\eta_{b_2})$$

where b_{b_1} is a solution of (6.15) satisfying $b_{b_1}(-l) = b_1$. Note that we have $b_{b_1}(l) = b_2$ if and only if $(b_1, b_2) \in \text{Graph } P^\dagger$. And the second map j is given by

$$j(\eta_-, b, \eta_+) = (\eta_- + \eta_{b(-l)}, \eta_+ - \eta_{b(l)}).$$

and so $\ker E^\dagger(u)$ has its dimension given by

$$\begin{aligned} &\dim \ker(D_{u_+} \bar{\partial}_{(K_+, J_+)})^\dagger + \dim \ker(D_{u_-} \bar{\partial}_{(K_-, J_-)})^\dagger + \dim \text{Graph } P^\dagger \cap (V_+^\perp \times V_-^\perp) \\ &= \dim \ker(D_{u_+} (\bar{\partial}_{(K_+, J_+)})^\dagger + \dim \ker(D_{u_-} \bar{\partial}_{(K_-, J_-)})^\dagger + \dim(P^\dagger \cdot V_-^\perp \cap V_+^\perp) \quad (6.17) \end{aligned}$$

Equivalently $E(u)$ has a closed range and its coker $E(u)$ has dimension the same as this. Combining this dimension counting of coker $E(u)$ with that of $\ker E(u)$ in (6.12), we conclude that $E(u)$ is Fredholm and has index given by

$$\begin{aligned} \text{Index } E(u) &= \left(\left(\dim \ker D_{u_+} \bar{\partial}_{(K_+, J_+)} + \dim \ker D_{u_-} \bar{\partial}_{(K_-, J_-)} - \dim(P \cdot V_- + V_+) \right) \right. \\ &\quad \left. - \left(\left(\dim \ker(D_{u_+} \bar{\partial}_{(K_+, J_+)})^\dagger + \dim \ker(D_{u_-} \bar{\partial}_{(K_-, J_-)})^\dagger + \dim(P^\dagger \cdot V_-^\perp \cap V_+^\perp) \right) \right) \right) \\ &= \text{Index } D_{u_+} \bar{\partial}_{(K_+, J_+)} + \text{Index } D_{u_-} \bar{\partial}_{(K_-, J_-)} - 2n \\ &= (n + \mu_{H_-}([z_-, w_-]) + 2c_1(A_-)) + (n - \mu_{H_+}([z_+, w_+]) + 2c_1(A_+)) \\ &= \mu_{H_-}([z_-, w_-]) - \mu_{H_+}([z_+, w_+]) + c_1(A_-) + c_1(A_+). \end{aligned}$$

Here we have used

$$\begin{aligned} &\dim(P \cdot V_- + V_+) + \dim(P^\dagger \cdot V_-^\perp \cap V_+^\perp) \\ &= \dim(P \cdot V_- + V_+) + \dim((P \cdot V_-)^\perp \cap V_+^\perp) \\ &= \dim(P \cdot V_- + V_+) + \dim(P \cdot V_- + V_+)^\perp = 2n, \end{aligned}$$

for the second identity and

$$\begin{aligned} \text{Index } D_{u_-} \bar{\partial}_{(K_-, J_-)} &= (n + \mu_{H_-}([z_-, w_-]) + 2c_1(A_-)) \\ \text{Index } D_{u_+} \bar{\partial}_{(K_+, J_+)} &= (n - \mu_{H_+}([z_+, w_+]) + 2c_1(A_+)) \end{aligned}$$

for the third identity.

To justify that $P^\dagger \cdot V_-^\perp = (P \cdot V_-)^\perp$: For convenience we let

$$\begin{aligned} P &: T_{u_-(o_-)} M \rightarrow T_{u_+(o_+)} M, \quad a(-l) \rightarrow a(l) \\ P^\dagger &: T_{u_-(o_-)} M \rightarrow T_{u_+(o_+)} M, \quad b(-l) \rightarrow b(l) \end{aligned}$$

be the linear maps for solutions a and b of ODE (6.11) and (6.15) respectively. Then for solutions a, b we have

$$\frac{d}{d\tau} \langle a(\tau), b(\tau) \rangle = \langle \nabla \text{grad } f(\chi) a(\tau), b(\tau) \rangle + \langle a(\tau), -\nabla \text{grad } f(\chi) b(\tau) \rangle = 0,$$

This in particular implies

$$\langle Pa, P^\dagger b \rangle = \langle a(l), b(l) \rangle = \langle a(-l), b(-l) \rangle = \langle a, b \rangle.$$

and hence

$$P^\dagger V_-^\perp = (PV_-)^\perp.$$

□

Corollary 6.3. *Suppose $u_\pm \in \mathcal{M}([z_\pm, w_\pm]; A_\pm)$ are Fredholm regular, then $u \in \mathcal{M}([z_-, w_-]; f; [z_+, w_+]; A_\pm)$ is Fredholm regular (in the sense that $E(u)$ is surjective) if and only if the configuration $u = (u_-, \chi, u_+)$ satisfies the “disk-flow-disk” transversality in definition 1.1.*

Proof. In (6.17) of the above proposition we have obtained

$$\begin{aligned} \dim \ker E^\dagger(u) &= \dim \ker (D_{u_+}(\bar{\partial}_{(K_+, J_+)})^\dagger + \dim \ker (D_{u_-} \bar{\partial}_{(K_-, J_-)})^\dagger \\ &\quad + \dim (P^\dagger \cdot V_-^\perp \cap V_+^\perp) \\ &= \dim (P^\dagger \cdot V_-^\perp \cap V_+^\perp) \\ &= \dim (PV_- + V_+)^\perp. \end{aligned}$$

Hence $E(u)$ is surjective if and only if $(PV_- + V_+)^\perp = \{0\}$, i.e.

$$PV_- + V_+ = T_{u_+(o_+)}M.$$

But this is equivalent to

$$d\phi_f^{2l} ev_- \times ev_+ \pitchfork \Delta_{u_+(o_+)}^{tn},$$

as derived in (6.12). □

We then show $E(u)$ is surjective for generic f , for any $u = (u_-, \chi, u_+) \in \mathcal{M}([z_-, w_-]; f; [z_+, w_+]; A_\pm)$ of any $l > 0$. We need the variation of l to get this surjectivity. More precisely we have the following

Proposition 6.4. *Suppose that $u_\pm \in \mathcal{M}(K_\pm, J_\pm; [z_\pm, w_\pm]; A_\pm)$ is Fredholm regular, i.e., its linearization is surjective. Then for each given $l_0 > 0$, there exists a dense subset of $f \in C^\infty(M)$ such that any element $u = (u_-, \chi, u_+, l)$ in*

$$\mathcal{M}^{para}([z_-, w_-]; f; [z_+, w_+]; A_\pm) = \bigcup_{l \geq l_0} \mathcal{M}^l([z_-, w_-]; f; [z_+, w_+]; A_\pm)$$

is Fredholm regular, in the sense that $E(u)$ is surjective. $\mathcal{M}^{para}([z_-, w_-]; f; [z_+, w_+]; A_\pm)$ is a smooth manifold with dimension equal to the index of $E(u)$:

$$\begin{aligned} &\dim \mathcal{M}^{para}([z_-, w_-]; f; [z_+, w_+]; A_\pm) \\ &= \mu_{H_-}([z_-, w_-]) - \mu_{H_+}([z_+, w_+]) + c_1(A_-) + c_1(A_+) + 1. \end{aligned}$$

In particular when $A_- + A_+ = 0$ in $\pi_2(M)$, the index becomes

$$\mu_{H_-}([z_-, w_-]) - \mu_{H_+}([z_+, w_+]) + 1. \quad (6.18)$$

Proof. Consider the section e of the following Banach bundle

$$\begin{aligned} e &: \mathcal{B}^{dfd}(z_-, z_+) \times C^\infty(M) \rightarrow \mathcal{L}^{dfd}(z_-, z_+) \\ e &: (u, f) \rightarrow (\bar{\partial}_{(K_-, J_-)} u_-, \dot{\chi} - \nabla f(\chi), \bar{\partial}_{(K_+, J_+)} u_+). \end{aligned}$$

where $u = (u_-, \chi, u_+, l) \in \mathcal{B}_l^{dfd}(z_-, z_+) \subset \mathcal{B}^{dfd}(z_-, z_+)$. Denote the linearization of e at (u, f) by

$$E(u, f) := D_{(u, f)} e : T_u \mathcal{B}^{dfd}(z_-, z_+) \times T_f C^\infty(M) \rightarrow L_u^p(z_-, z_+).$$

For $\xi_\pm \in W^{1,p}(u_\pm^* TM)$, $a \in W^{1,p}(\chi^* TM)$, $\mu \in T_l \mathbb{R}$ and $h \in T_f C^\infty(M)$,

$$E(u, f) : (\xi_-, a, \xi_+, \mu, h) \rightarrow \eta := (\eta_-, b, \eta_+)$$

where

$$\begin{aligned} \eta_- &= D_{u_-} \bar{\partial}_{(K_-, J_-)} \xi_-, \\ b &= \frac{D}{d\tau} a - \nabla_a \text{grad}(f) + \frac{\mu}{l} \dot{\chi}(\tau) - \nabla h(\chi), \\ \eta_+ &= D_{u_+} \bar{\partial}_{(K_+, J_+)} \xi_+. \end{aligned}$$

Next we show the cokernel of $E(u, f)$ vanishes. Let $E(u, f)^*$ be the adjoint operator of $E(u, f)$, such that

$$E(u, f)^* : L_u^p(z_-, z_+)^* \rightarrow (W^{1,p}(z_-, z_+; dfd) \times T_f C^\infty(M))^*.$$

Using the nondegenerate L^2 pairing

$$L^p(\Lambda^{(0,1)} u^* TM) \times L^q(\Lambda^{(1,0)} u^* TM) \rightarrow \mathbb{R}$$

we identify $L_u^p(z_-, z_+)^*$ with $L^q(\Lambda^{(1,0)} u^* TM)$. On the other hand, we can identify $(W^{1,p}(z_-, z_+; dfd) \times T_f C^\infty(M))^*$ with $W^{-1,q}(z_-, z_+; dfd) \times (T_f C^\infty(M))^*$ where $W^{-1,q}(z_-, z_+; dfd)$ is defined to be

$$\{(\xi_-, a, \xi_+, \mu) \in W^{1,p}(z_-, z_+) \mid \xi_-(o_-) = a(-l) - \mu \dot{\chi}(-l), \xi_+(o_+) = a(l) + \mu \dot{\chi}(l)\}^\perp$$

in the direct product

$$W^{-1,q}(z_-, z_+) = W^{-1,q}(u_-^* TM) \times W^{-1,q}(\chi^* TM) \times W^{-1,q}(u_+^* TM)$$

where $(\cdot)^\perp$ denotes the L^2 -orthogonal complement. Here we have $1 < q < 2$ since $2 < p < \infty$.

We denote by

$$E(u, f)^\dagger : L^q(\Lambda^{(1,0)} u^* TM) \rightarrow W^{-1,q}(z_-, z_+; dfd) \times (T_f C^\infty(M))^*$$

the corresponding L^2 -adjoint with respect to these identifications. Recall by definition, we have

$$\langle E(u, f) \xi, \eta \rangle = \langle \xi, E(u, f)^\dagger \eta \rangle.$$

For any given $\eta := (\eta_-, b, \eta_+) \in \ker E^\dagger(u, f) \subset L^q(\Lambda^{(1,0)} u^* TM)$, it satisfies

$$\langle E(u, f) \xi, \eta \rangle = 0$$

for all $(\xi_+, a, \xi_-, \mu) \in T_u \mathcal{B}_l^{dfd}(z_-, z_+)$, especially for $(\xi_+, a, \xi_-, 0) \in T_u \mathcal{B}_l^{dfd}(z_-, z_+)$, namely

$$\begin{aligned} 0 &= \int_{-l}^l \left\langle \frac{Da}{d\tau} - \nabla \text{grad} f(\chi) a, b \right\rangle - \int_{-l}^l \langle \nabla h(\chi), b \rangle \\ &\quad + \int_{\Sigma_-} \langle D_{u_-} \bar{\partial}_{(K_-, J_-)} \xi_-, \eta_- \rangle + \int_{\Sigma_+} \langle D_{u_+} \bar{\partial}_{(K_+, J_+)} \xi_+, \eta_+ \rangle \end{aligned} \quad (6.19)$$

for all the triples satisfying the matching condition

$$\xi_-(o_-) = a(-l), \quad \xi_+(o_+) = a(l). \quad (6.20)$$

Letting $a = \xi_- = \xi_+ = 0$, then (6.19) becomes

$$\int_{-l}^l \langle \nabla h(\chi), b \rangle = 0$$

for all $h \in C^\infty(M)$, so $b = 0$. Now (6.19) becomes

$$\int_{\dot{\Sigma}_-} \langle D_{u_-} \bar{\partial}_{(K^-, J_-)} \xi_-, \eta_- \rangle + \int_{\dot{\Sigma}_+} \langle D_{u_+} \bar{\partial}_{(K_+, J_+)} \xi_+, \eta_+ \rangle = 0.$$

Notice that ξ_- and ξ_+ can vary independently since $a(-l)$ and $a(l)$ can be any vector without restriction and hence the matching condition (6.20) does not put any restriction on $\xi_\pm(o_\pm)$. Therefore we have

$$\begin{aligned} \int_{\dot{\Sigma}_-} \langle D_{u_-} \bar{\partial}_{(K^-, J_-)} \xi_-, \eta_- \rangle &= 0, \\ \int_{\dot{\Sigma}_+} \langle D_{u_+} \bar{\partial}_{(K_+, J_+)} \xi_+, \eta_+ \rangle &= 0 \end{aligned}$$

for all ξ_\pm . In other words, η_\pm lie in $\text{coker } D_{u_\pm} \bar{\partial}_{(K_\pm, J_\pm)}$. By the hypothesis, we conclude $\eta_\pm = 0$. Therefore the cokernel of $E(u, f)$ vanishes in $E(u, f) : W^{1,p} \rightarrow L^p$ setting.

Now we raise the regularity to $W^{k,p}$ setting. For any $k \geq 1$ and $\zeta \in W^{k-1,p} \subset L^p$, from vanishing of the above cokernal we can always solve $E(u, f)\xi = \zeta$ with $\xi \in W^{1,p}$. By ellipticity of $E(u, f)$ we have $\xi \in W^{k,p}$. Therefore, the cokernal of $E(u, f)$ vanishes if we treat $E(u, f)$ as a map from $W^{k,p}$ to $W^{k-1,p}$ for any k . We fix a large enough k (which depends on $n, c_1(A_\pm), \mu_{H_\pm}([z_\pm, w_\pm])$) to define the $W^{k,p}$ Banach norm on $\mathcal{B}^{dfd}(z_-, z_+)$. Therefore, the universal moduli space

$$\mathcal{M}_{univ}^{para}([z_-, w_-]; f; [z_+, w_+]; A_\pm) = e^{-1}(0)$$

is a $W^{k,p}$ Banach manifold in $\mathcal{B}^{dfd}(z_-, z_+) \times C^\infty(M)$. For the natural projection

$$\pi : \mathcal{M}_{univ}^{para}([z_-, w_-]; f; [z_+, w_+]; A_\pm) \rightarrow C^\infty(M),$$

since our k is large enough, we can apply Sard-Smale theorem and conclude its regular values f form a set of second category in $C^\infty(M)$. For any regular value f ,

$$\mathcal{M}_{univ}^{para}([z_-, w_-]; f; [z_+, w_+]; A_\pm) = \pi^{-1}(f) \cap \mathcal{M}_{univ}^{para}([z_-, w_-]; f; [z_+, w_+]; A_\pm)$$

is a finite dimensional smooth submanifold with

$$\begin{aligned} &\dim \mathcal{M}_{univ}^{para}([z_-, w_-]; f; [z_+, w_+]; A_\pm) \\ &= \text{index}(\pi) = \text{index}(E(u)) \\ &= \mu_{H_-}([z_-, w_-]) - \mu_{H_+}([z_+, w_+]) + c_1(A_-) + c_1(A_+) + 1. \end{aligned}$$

The $\text{index } E(u)$ here differs from previous Proposition by 1 because the linearization $E(u)$ here is in $T_u \mathcal{B}^{dfd}(z_-, z_+)$ instead of $T_u \mathcal{B}_l^{dfd}(z_-, z_+)$. Since f is a regular value of π , $E(u) : T_u \mathcal{B}^{dfd}(z_-, z_+) \rightarrow \mathcal{L}^{dfd}(z_-, z_+)$ is surjective and u is Fredholm regular. \square

Next we establish the condition to ensure the joint points $u_\pm(o_\pm)$ are immersed. This condition satisfies for a generic choice of almost complex structures J and will be needed in the proof of the surjectivity of our gluing: The Hausdorff convergence imposed in Definition 4.2 (2) is not strong enough to detect multiple covers of the thin cylinder although it captures all simple thin cylinders. Immersion condition

at the joint points then make the thin part of the adiabatic limit with immersed joint points be automatically simple.

Proposition 6.5. *For generic $J_0 \in \mathcal{J}_\omega$ and $f \in C^\infty(M)$, any element $u = (u_-, \chi, u_+, l) \in \mathcal{M}^{para}([z_-, w_-]; f; [z_+, w_+]; A_\pm)$ whose u_\pm are somewhere injective must be Fredholm regular, and have both $u_\pm(o_\pm)$ immersed, provided that*

$$\mu_{H_-}([z_-, w_-]) - \mu_{H_+}([z_+, w_+]) + 2c_1(A_-) + 2c_1(A_+) < 2n - 1,$$

or equivalently

$$\dim \mathcal{M}^{para}([z_-, w_-]; f; [z_+, w_+]; A_\pm) \leq 2n - 1.$$

Epecially, for any $\mathcal{M}^{para}([z_-, w_-]; f; [z_+, w_+]; A_\pm)$ of virtual dimension 0, 1, both $u_\pm(o_\pm)$ are immersed for generic (J_0, f) .

Proof. The proof is a small variant of Theorem 1.2 of [OZ2]. Consider the section Υ of the following Banach bundle

$$\begin{aligned} \Upsilon &: \mathcal{B}^{dfd}(z_-, z_+) \times C^\infty(M) \times \mathcal{J}_\omega \rightarrow \mathcal{L}^{dfd}(z_-, z_+) \times H_-^{1,0} \times H_+^{1,0}, \\ \Upsilon &: (u, f, J_0) \rightarrow (\bar{\partial}_{(K^-, J_-)} u_-, \dot{\chi} - \nabla f(\chi), \bar{\partial}_{(K^+, J_+)} u_+, \partial_{J_0} u_-(o_-), \partial_{J_0} u_+(o_+)) \end{aligned}$$

where

$$H_\pm^{0,1}(u_\pm) = \text{Hom}_{i, J_0}(T_{o_\pm} \Sigma_\pm, T_{u_\pm(o_\pm)} M)$$

is a rank $2n$ vector bundle over $\mathcal{B}^{dfd}(z_-, z_+) \times C^\infty(M) \times \mathcal{J}_\omega$ whose fiber consists of (j, J_0) -linear maps from $T_{o_\pm} \Sigma_\pm$ to $T_{u_\pm(o_\pm)} M$. Here the j for $\Sigma_\pm \simeq (D^2, o_\pm)$ is fixed. Any cokernal element

$$(\eta_-, b, \eta_+, \alpha_-, \alpha_+) \in \mathcal{L}^{dfd}(z_-, z_+) \times H_-^{1,0} \times H_+^{1,0}$$

of $D_{(u, f, J_0)} \Upsilon$ must satisfy

$$\begin{aligned} 0 &= \int_{-l}^l \left\langle \frac{Da}{\partial \tau} - \nabla \text{grad } f(\chi) a, b \right\rangle - \int_{-l}^l \langle \nabla h(\chi), b \rangle \\ &\quad + \int_{\dot{\Sigma}_-} \left\langle D_{u_-} \bar{\partial}_{(K^-, J_-)} \xi_- + \frac{1}{2} du_- \circ B \circ j, \eta_- \right\rangle \\ &\quad + \int_{\dot{\Sigma}_+} \left\langle D_{u_+} \bar{\partial}_{(K^+, J_+)} \xi_+ + \frac{1}{2} du_+ \circ B \circ j, \eta_+ \right\rangle \\ &\quad + \langle D_{u_-} \partial_{(K^-, J_-)} \xi_-, \alpha_- \delta_{o_-} \rangle_{\dot{\Sigma}_-} + \langle D_{u_+} \partial_{(K^+, J_+)} \xi_+, \alpha_+ \delta_{o_+} \rangle_{\dot{\Sigma}_+} \quad (6.21) \end{aligned}$$

for all $(\xi_+, a, \xi_-, \mu) \in T_u \mathcal{B}^{dfd}(z_-, z_+)$, $h \in T_f C^\infty(M)$ and $B \in T_{J_0} \mathcal{J}_\omega$, where δ_{o_\pm} are delta functions at o_\pm on $\dot{\Sigma}_\pm$.

Letting $a = \xi_- = \xi_+ = B = 0$, then (6.21) becomes

$$\int_{-l}^l \langle \nabla h(\chi), b \rangle = 0$$

for all $h \in C^\infty(M)$, so $b = 0$. Now (6.21) becomes

$$\begin{aligned} 0 &= \int_{\dot{\Sigma}_-} \left\langle D_{u_-} \bar{\partial}_{(K^-, J_-)} \xi_- + \frac{1}{2} du_- \circ B \circ j, \eta_- \right\rangle + \langle D_{u_-} \partial_{(K^-, J_-)} \xi_-, \alpha_- \delta_{o_-} \rangle_{\dot{\Sigma}_-} \\ &\quad + \int_{\dot{\Sigma}_+} \left\langle D_{u_+} \bar{\partial}_{(K^+, J_+)} \xi_+ + \frac{1}{2} du_+ \circ B \circ j, \eta_+ \right\rangle + \langle D_{u_+} \partial_{(K^+, J_+)} \xi_+, \alpha_+ \delta_{o_+} \rangle_{\dot{\Sigma}_+}. \end{aligned}$$

Notice that ξ_- and ξ_+ can vary independently since $a(-l)$ and $a(l)$ can be any vector without restriction and hence the matching condition (6.20) does not put any restriction on $\xi_{\pm}(o_{\pm})$. Therefore we have

$$\begin{aligned} \int_{\dot{\Sigma}_-} \left\langle D_{u_-} \bar{\partial}_{(K^-, J^-)} \xi_- + \frac{1}{2} du_- \circ B \circ j, \eta_- \right\rangle + \langle D_{u_-} \partial_{(K^-, J^-)} \xi_-, \alpha_- \delta_{o_-} \rangle_{\dot{\Sigma}_-} &= 0, \\ \int_{\dot{\Sigma}_+} \left\langle D_{u_+} \bar{\partial}_{(K_+, J_+)} \xi_+ + \frac{1}{2} du_+ \circ B \circ j, \eta_+ \right\rangle + \langle D_{u_+} \partial_{(K_+, J_+)} \xi_+, \alpha_+ \delta_{o_+} \rangle_{\dot{\Sigma}_+} &= 0, \end{aligned}$$

which are identical to (2.12) in [OZ2]. Then the remaining steps are the same as in [OZ2] to show $\eta_{\pm} = 0$ and $\alpha_{\pm} = 0$, hence Υ is transversal to the section zero sections

$$\begin{aligned} o_{\mathcal{L}^{df d}(z_-, z_+)} \times o_{H_-^{1,0}} \times H_+^{1,0} \text{ and} \\ o_{\mathcal{L}^{df d}(z_-, z_+)} \times H_-^{1,0} \times o_{H_+^{1,0}}, \end{aligned}$$

which both have codimension $2n$ in $o_{\mathcal{L}^{df d}(z_-, z_+)} \times H_-^{1,0} \times H_+^{1,0}$. Hence for generic (J_0, f) , by Sard Theorem as in [OZ2],

$$\Upsilon^{-1} \left(o_{\mathcal{L}^{df d}(z_-, z_+)} \times o_{H_-^{1,0}} \times H_+^{1,0} \cup o_{\mathcal{L}^{df d}(z_-, z_+)} \times H_-^{1,0} \times o_{H_+^{1,0}} \right)$$

has dimension $2n$ less than

$$\Upsilon^{-1} \left(o_{\mathcal{L}^{df d}(z_-, z_+)} \times H_-^{1,0} \times H_+^{1,0} \right) = \mathcal{M}^{para}([z_-, w_-]; f; [z_+, w_+]; A_{\pm}),$$

so is of dimension

$$\mu_{H_-}([z_-, w_-]) - \mu_{H_+}([z_+, w_+]) + 2c_1(A_-) + 2c_1(A_+) + 1 - 2n,$$

which is negative if the index condition of the proposition holds. But this means the set of the (u_-, χ, u_+) who fails the immersion condition at o_- or o_+ is empty. The proposition is proved. \square

7. J -HOLOMORPHIC CURVES FROM CYLINDRICAL DOMAIN

The (perturbed) J -holomorphic curves from punctured Riemannian surface $\dot{\Sigma}$ have been extensively studied in the literature. Since we will use cylindrical coordinates of punctured Riemannian surface in our gluing analysis, we briefly review the Banach norms in this setting. For our u_{\pm} , the domain (D^2, o_{\pm}) can be thought as

$$\Sigma_{\pm} = (D^2, o_{\pm}) \simeq [-\infty, \infty] \times S^1,$$

where $o_{\pm} \simeq \{-\pm\infty\} \times S^1$ and $\partial D^2 \simeq \{\pm\infty\} \times S^1$. Let $O_{\pm} \simeq -[\pm\infty, 0) \times S^1$ be the cylindrical neighborhood near the puncture $o_{\pm} = \{-\pm\infty\} \times S^1$. Without loss of generality let's assume the image of O_{\pm} under u_{\pm} has diameter less than the injective radius of (M, g) (otherwise we can shift the \mathbb{R} component to reparametrize u_{\pm}). One can use the following norm to define the Banach manifold hosting such Floer trajectories. Let

$$\mathcal{B}_+(z_{\pm}) = \left\{ u_{\pm} : \Sigma_{\pm} \rightarrow M \in W_{loc}^{1,p}, \left| \begin{aligned} &u_+(-\infty, t) = p_{\pm} \text{ for some } p_{\pm} \in M, \\ &u_+(\infty, t) = z_+(t), \\ &u_+(\tau, t) = \exp_{p_+} \xi(\tau, t) \text{ near } p_+, \\ &u_+(\tau, t) = \exp_{z_+(t)} \xi(\tau, t) \text{ near } z_+, \\ &\text{and } e^{\frac{2\pi\delta|\tau|}{p}} \xi \in W^{1,p}(\mathbb{R} \times S^1). \end{aligned} \right. \right\}.$$

We define \mathcal{B}_- similarly with ∞ and $-\infty$ switched around. For any section $\xi_\pm \in W_{loc}^{1,p}(u_\pm^* TM)$, we define its Banach norm to be

$$\|\xi_\pm\|_{W_\alpha^{1,p}(\Sigma_\pm)} := \|\widetilde{\xi}_\pm\|_{W_\delta^{1,p}(\Sigma_\pm)} + |\xi_\pm(o_\pm)|,$$

where $\widetilde{\xi}_\pm$ is defined by

$$\widetilde{\xi}_\pm = \xi_\pm - \varphi_\pm(\tau, t) Pal_\pm(\tau, t)(\xi_\pm(o_\pm))$$

and

$$\|\widetilde{\xi}_\pm\|_{W_\delta^{1,p}(\Sigma_\pm)} := \left\| e^{\frac{2\pi\delta|\tau|}{p}} \widetilde{\xi}_\pm \right\|_{W^{1,p}(\mathbb{R} \times S^1)}.$$

Here $\varphi_\pm(\tau, t)$ is a smooth cut-off function such that

$$\varphi_\pm(\tau, t) = \begin{cases} 1 & \text{on } \pm[-\infty, -1) \times S^1 \\ 0 & \text{outside the neighborhood of } O_\pm \end{cases}$$

and $|d\varphi_\pm| \leq 2$, and $Pal_\pm(\tau, t)$ is the parallel transport along the shortest geodesics from $u_\pm(o_\pm)$ to $u_\pm(\tau, t)$. This defines the Banach manifold structure on $\mathcal{B}_\pm(z_\pm)$. For any section $\eta_\pm \in \Gamma(u_\pm^* TM) \otimes \Lambda^{0,1}(\Sigma_\pm)$, we let

$$\|\eta\|_{L_\delta^p(\Sigma_\pm)} := \left\| e^{\frac{2\pi\delta|\tau|}{p}} \eta \right\|_{L^p(\mathbb{R} \times S^1)}$$

and denote the set of all such η with $\|\eta\|_{L_\delta^p(\Sigma_\pm)} < \infty$ by $L_\delta^p(\Gamma(u_\pm^* TM) \otimes \Lambda^{0,1}(\Sigma_\pm))$. Then we let

$$\mathcal{L}_\pm(z_\pm) = \bigcup_{u_\pm \in \mathcal{B}_\pm(z_\pm)} L_\delta^p(\Gamma(u_\pm^* TM) \otimes \Lambda^{0,1}(\Sigma_\pm))$$

be the Banach bundle over $\mathcal{B}_\pm(z_\pm)$. It is well-known that

$$\overline{\partial}_{(K_\pm, J_\pm)} : \mathcal{B}_\pm(z_\pm) \rightarrow \mathcal{L}_\pm(z_\pm)$$

is a smooth Fredholm section, and for generic (K_\pm, J_\pm) the Floer trajectory $u_\pm \in (\overline{\partial}_{(K_\pm, J_\pm)})^{-1}(0)$ are Fredholm regular and hence

$$D_{u_\pm} \overline{\partial}_{(K_\pm, J_\pm)} : T_{u_\pm} \mathcal{B}_\pm(z_\pm) \simeq W_\delta^{1,p}(\Gamma(u_\pm^* TM)) \oplus T_{p_\pm} M \rightarrow L_\delta^p(\Gamma(u_\pm^* TM) \otimes \Lambda^{0,1}(\Sigma_\pm))$$

is surjective and has bounded right inverse, denoted by Q_\pm . Here we recall

$$T_{u_\pm} \mathcal{B}_\pm(z_\pm) \simeq W_\delta^{1,p}(\Gamma(u_\pm^* TM)) \oplus T_{p_\pm} M.$$

For the simplicity of notations in later calculation, we denote by

$$v_\pm := \varphi_\pm(\tau, t) Pal_\pm(\tau, t)(\xi_\pm(o_\pm)) \quad (7.1)$$

the cut-off *constant vector field* extending the vector $\xi_\pm(o_\pm)$.

Remark 7.1. With obvious modifications, we can use $\mathcal{B}_\pm(z_\pm)$ (using cylindrical measure) instead of $W^{1,p}(\dot{\Sigma}_\pm, M, z_\pm)$ (using compact measure) in last section to define the Banach manifold $\mathcal{B}^{df,d}(z_-, z_+)$. The index of $E(u)$ is the same, and $E(u)$ is surjective for generic (J, f) . Proofs are similar so omitted.

8. APPROXIMATE SOLUTIONS

In this section, we construct an approximate solution u_{app}^ε of the Floer equation. For the notational simplicity, we denote

$$\tau(\varepsilon) = \frac{l}{\varepsilon} + \frac{p-1}{\delta} S(\varepsilon) \quad (8.1)$$

which will appear very often in the discussion henceforth. We also denote the translation map

$$I_{\tau_0} : \mathbb{R} \times S^1 \rightarrow \mathbb{R} \times S^1, \quad I_{\tau_0}(\tau)(\tau, t) = (\tau - \tau_0, t) \quad (8.2)$$

for $\tau_0 \in \mathbb{R}$. To simplify the notations, we also denote

$$E(x, y) = (\exp_x)^{-1}(y) \quad (8.3)$$

whenever $d(x, y) < \iota_M$ where ι_M is the injectivity radius of (M, g) .

We recall the decomposition of \mathbb{R} into

$$-\infty < -\tau(\varepsilon) - 1 < -\tau(\varepsilon) < -R(\varepsilon) < R(\varepsilon) < \tau(\varepsilon) < \tau(\varepsilon) + 1 < \infty$$

we made from the beginning of section 3 where we made a choice

$$R(\varepsilon) = \frac{l}{\varepsilon}, \quad \tau(\varepsilon) = R(\varepsilon) + \frac{1}{2\pi} \ln \left(1 + \frac{l}{\varepsilon} \right).$$

And we denoted $K_\pm(\tau, t, x) = \kappa^\pm(\tau) H_t(x)$. Note that this latter representation of K_\pm depend on the choice of analytic coordinates (τ, t) compatible to the parameter t parameterizing H_t near the punctures respectively. The coordinates are unique modulo translations by τ .

Now let u_\pm be solutions of the equations $(du + P_{K_\pm})_{J_\pm}^{(0,1)} = 0$ respectively and fix the coordinate representations of $u_\pm = u_\pm(\tau, t)$ so that they are compatible with the choice of analytic coordinates given at the punctures. This can be always done by adjusting the choice of analytic coordinates near the two punctures (e_+, o_+) and (e_-, o_-) respectively.

With this preparation, we define our approximate solution by

$$u_{app}^\varepsilon(\tau, t) = \begin{cases} u_-(\tau + \tau(\varepsilon), t) & -\infty < \tau \leq -l/\varepsilon - 1 \\ \exp_{\chi(\varepsilon\tau)} \left[(1 - \kappa_\varepsilon^0(\tau)) E(\chi(\varepsilon\tau), u_-^\varepsilon(\tau, t)) \right] & -l/\varepsilon - 1 \leq \tau \leq -l/\varepsilon \\ \chi(\varepsilon\tau) & -l/\varepsilon \leq \tau \leq l/\varepsilon, \\ \exp_{\chi(\varepsilon\tau)} \left[(1 - \kappa_\varepsilon^0(\tau)) E(\chi(\varepsilon\tau), u_+^\varepsilon(\tau, t)) \right] & l/\varepsilon \leq \tau \leq l/\varepsilon + 1 \\ u_+(\tau - \tau(\varepsilon), t) & l/\varepsilon + 1 \leq \tau < \infty \end{cases} \quad (8.4)$$

where

$$\begin{aligned} u_-^\varepsilon(\tau, t) &= u_-(\tau + \tau(\varepsilon), t), \\ u_+^\varepsilon(\tau, t) &= u_+(\tau - \tau(\varepsilon), t), \\ S(\varepsilon) &= \frac{1}{2\pi} \ln(1 + l/\varepsilon), \end{aligned}$$

and $\kappa_\varepsilon^0(\tau)$ is a smooth cut-off functions defined in (3.9), $\kappa_\varepsilon^0(\tau) = 1$ when $|\tau| \leq l/\varepsilon$ and $\kappa_\varepsilon^0(\tau) = 0$ when $|\tau| \geq l/\varepsilon + 1$.

The assignment of the approximate solution to each (u_-, χ, u_+) and $0 < \varepsilon < \varepsilon_0$ defines a smooth map

$$\begin{aligned} \text{preG} : \mathcal{M}_1(K_-, z_-; A_-)_{ev_+} \times_{ev_0} \mathcal{M}(f; [0, \ell])_{ev_\ell} \times_{ev_-} \mathcal{M}_1(K_+, z_+; A_+) \times (0, \varepsilon_0] \\ \rightarrow \bigcup_{0 < \varepsilon \leq \varepsilon_0} \mathcal{M}^\varepsilon(K_\varepsilon; z_-, z_+; B). \end{aligned} \quad (8.5)$$

Later in carrying out the $\bar{\partial}_{(K_\varepsilon, J_\varepsilon; \varepsilon f)}$ -error estimate, we make some simplification of the expression of u_{app}^ε . Notice that the interpolation in u_{app}^ε takes place in a ball of radius $C\varepsilon$ around p_\pm , since

$$\begin{aligned} \sup_{l/\varepsilon \leq |\tau| \leq l/\varepsilon + 1} \text{dist}(\chi_\varepsilon(\tau), p_\pm) &= \sup_{l/\varepsilon \leq |\tau| \leq l/\varepsilon + 1} \text{dist}(\chi_\varepsilon(\tau), \chi_\varepsilon(\pm l/\varepsilon)) \\ &\leq \varepsilon \sup |\nabla f| \leq C\varepsilon \end{aligned} \quad (8.6)$$

and

$$\begin{aligned} \sup_{l/\varepsilon \leq |\tau| \leq l/\varepsilon + 1} \text{dist}(u_\pm^\varepsilon(\tau), p_\pm) &= \sup_{l/\varepsilon \leq |\tau| \leq l/\varepsilon + 1} \text{dist}(u_\pm^\varepsilon(\tau), u_\pm^\varepsilon(\pm\infty)) \\ &\leq Ce^{-2\pi \frac{p-1}{\delta} S(\varepsilon)} = C(1 + l/\varepsilon)^{-\frac{p-1}{\delta}} \end{aligned} \quad (8.7)$$

$$\leq \tilde{C}(l)\varepsilon, \quad (8.8)$$

where the first inequality is because we have

$$|u_\pm^\varepsilon(\tau, t) - u_\pm^\varepsilon(\pm\infty, t)| \leq Ce^{-2\pi|\tau - \pm(\tau(\varepsilon))|}$$

by the J -holomorphic property of u_\pm and the second inequality is because we have chosen $\delta < p - 1$ and set $\tilde{C}(l) = Cl^{-\frac{p-1}{\delta}}$.

At the interpolation $|\tau| \in [l/\varepsilon, l/\varepsilon + 1]$, both $u_\pm^\varepsilon(\tau, t)$ and $\chi(\varepsilon\tau)$ are in the ball of radius $C\varepsilon$ around p_\pm . The expression

$$\exp_{\chi(\varepsilon\tau)} [(1 - \kappa_\varepsilon^0(\tau)) E(\chi(\varepsilon\tau), u_\pm^\varepsilon(\tau, t))] =: v_\varepsilon^\pm(\tau, t)$$

in (8.4) is C^1 close to $\chi(\varepsilon\tau) + (1 - \kappa_\varepsilon^0(\tau)) (u_\pm^\varepsilon(\tau, t) - \chi(\varepsilon\tau))$, i.e.

$$\kappa_\varepsilon^0(\tau) \chi(\varepsilon\tau) + (1 - \kappa_\varepsilon^0(\tau)) u_\pm^\varepsilon(\tau, t) =: \tilde{v}_\varepsilon^\pm(\tau, t)$$

in coordinates, where the $+$ is from the vector space structure from $T_{p_\pm}M$, and the C^1 difference is of order $C\varepsilon$. To see this, we identify the geodesic ball of radius $C\varepsilon$ around p_\pm to the ball in $T_{p_\pm}M$, and equip it with the Euclidean metric g_{p_\pm} . If we deform the metric involved in the exponential map in the expression

$$\exp_{\chi(\varepsilon\tau)} [(1 - \kappa_\varepsilon^0(\tau)) E(\chi(\varepsilon\tau), u_\pm^\varepsilon(\tau, t))]$$

to the Euclidean metric g_{p_\pm} , the result becomes $\kappa_\varepsilon^0(\tau) \chi(\varepsilon\tau) + (1 - \kappa_\varepsilon^0(\tau)) u_\pm^\varepsilon(\tau, t)$. Since the geodesic equation is a second order differential equation whose coefficients are polynomials on metric g and its first order derivatives, by the differentiability of solutions of ODE on finite interval $[0, 1]$ with respect to its initial condition and parameter, we see the C^1 norm of $\exp_{\chi(\varepsilon\tau)} [(1 - \kappa_\varepsilon^0(\tau)) \exp_{\chi(\varepsilon\tau)}^{-1}(u_\pm^\varepsilon(\tau, t))]$

depends on the C^1 norm of the section $g \in \Gamma(\text{Sym}_+(M))$ with bounded Lipschitz constant, where $\text{Sym}_+(M)$ is the space of positive definite symmetric tensors on

M . Since the C^1 difference of the metrics g_{p_\pm} and g is of order $C\varepsilon$ inside such ball, we have

$$\begin{aligned} \text{dist}(v_\varepsilon^\pm(\tau, t), \tilde{v}_\varepsilon^\pm(\tau, t)) &\leq C\varepsilon, \\ \|\nabla v_\varepsilon^\pm(\tau, t) - \text{Pal}\nabla\tilde{v}_\varepsilon^\pm(\tau, t)\| &\leq C\varepsilon, \end{aligned}$$

where Pal is the parallel transport along short geodesic from $\tilde{v}_\varepsilon^\pm(\tau, t)$ to $v_\varepsilon^\pm(\tau, t)$.

After the above simplification, we use the more explicit approximate solution

$$u_{app}^\varepsilon(\tau, t) = \begin{cases} u_-(\tau + \tau(\varepsilon), t) & -\infty < \tau \leq -l/\varepsilon \\ \kappa_\varepsilon^0(\tau) \chi(\varepsilon\tau) + (1 - \kappa_\varepsilon^0(\tau)) u_-^\varepsilon(\tau, t) & -l/\varepsilon - 1 \leq \tau \leq -l/\varepsilon \\ \chi(\varepsilon\tau) & -l/\varepsilon \leq \tau \leq l/\varepsilon \\ \kappa_\varepsilon^0(\tau) \chi(\varepsilon\tau) + (1 - \kappa_\varepsilon^0(\tau)) u_+^\varepsilon(\tau, t) & l/\varepsilon \leq \tau \leq l/\varepsilon + 1 \\ u_+(\tau - \tau(\varepsilon), t) & l/\varepsilon + 1 \leq \tau < \infty \end{cases} \quad (8.9)$$

where the vector sum $+$ is from the linear space structure of $T_{p_\pm}M$.

Remark 8.1. Apparently there enters no local model inserted at the joint points p_\pm to smooth out the join points in the construction of the above approximate solution. Implicitly there is, though. The local model at p_\pm is $u_\pm^{lmd} : \mathbb{R} \times S^1 \rightarrow \mathbb{C}^n \simeq (T_{p_\pm}M, J_{p_\pm})$,

$$u_\pm^{lmd}(\tau, t) = A_\pm z + a_\pm \tau,$$

where $z = e^{2\pi(\tau+it)}$, $A_\pm = u'_\pm(o_\pm)$, and $a_\pm = \nabla f(p_\pm)$. Then one can see the local model is the linearized version of the above intropolation of u_\pm and χ in u_{app}^ε . Because we can identify the portions of the approximate solution to u_\pm and χ respectively and borrow Fredholm theories there, we do not need to develop a Fredholm theory of the local model. However, when the gradient flow length is 0, the Fredholm theory of local model is needed for gluing, because during compactification to nodal Floer trajectories the information of ∇f is lost (See [OZ1]).

9. OFF-SHELL FORMULATION OF RESOLVED FLOER TRAJECTORIES FOR $\varepsilon > 0$

We define the Banach manifold to host resolved Floer trajectories near the “disk-flow-disk” Floer trajectories $(u_-, (\chi, l), u_+)$. The precise description is in order.

First we define the Banach manifold $\mathcal{B}_{res}^\varepsilon = \mathcal{B}_{res}^\varepsilon(z_-, z_+; l)$ for any $\varepsilon \in (0, \varepsilon_0)$ and $l \in (0, \infty)$, where $\varepsilon_0 > 0$ is a small constant to be determine later. To define \mathcal{B}_ε that hosts resolved Floer trajectories out of (u_-, χ, l, u_+) , we define the weighting function $\beta_{\delta, \varepsilon}$ as the gluing of the power weight and the exponential weight:

$$\beta_{\delta, \varepsilon}(\tau) = \begin{cases} 1 & \tau < -\tau(\varepsilon) \\ e^{2\pi\delta(\tau+\tau(\varepsilon))} & -\tau(\varepsilon) \leq \tau \leq -l/\varepsilon \\ \kappa_\varepsilon^0(\tau) \varepsilon^{1-p+\delta} (1 + |\tau|)^\delta + (1 - \kappa_\varepsilon^0(\tau)) e^{2\pi\delta(\tau+\tau(\varepsilon))} & -l/\varepsilon - 1 \leq \tau \leq -l/\varepsilon \\ \varepsilon^{1-p+\delta} (1 + |\tau|)^\delta & -l/\varepsilon \leq \tau \leq l/\varepsilon \\ \kappa_\varepsilon^0(\tau) \varepsilon^{1-p+\delta} (1 + |\tau|)^\delta + (1 - \kappa_\varepsilon^0(\tau)) e^{2\pi\delta(-\tau+\tau(\varepsilon))} & l/\varepsilon \leq \tau \leq l/\varepsilon + 1 \\ e^{2\pi\delta(-\tau+\tau(\varepsilon))} & l/\varepsilon + 1 \leq \tau \leq \tau(\varepsilon) \\ 1 & \tau > \tau(\varepsilon) \end{cases}$$

where $\kappa_\varepsilon^0(\tau)$ is the smooth cut-off function defined in (3.9) such that $\kappa_\varepsilon^0(\tau) = 1$ for $|\tau| \leq l/\varepsilon$ and $\kappa_\varepsilon^0(\tau) = 0$ for $|\tau| \geq l/\varepsilon + 1$. Note that $\beta_{\delta, \varepsilon}(\tau)$ is no less than

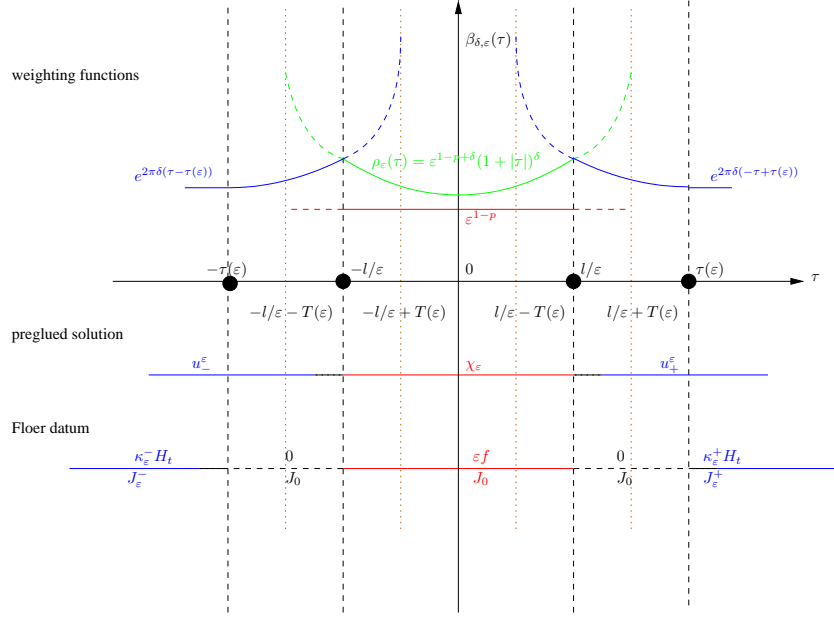


FIGURE 1. Weighting functions

1 everywhere, which is important for the uniform Sobolev constant we will discuss later in Section 13.

In the above figure we put the graphs of various weighting functions together, where the higher constant weight ϵ^{1-p} in the adiabatic weight $\|\cdot\|_{W_\epsilon^{1,p}}$ is in red horizontal line, power order weight $\rho_\epsilon(\tau)$ is in green, and the exponential weight is in blue. The weight $\beta_{\delta,\epsilon}(\tau)$ is the glue of the power weight and the exponential weight with smoothing at the corners, but to avoid too many graphs in the picture we did not draw the smoothing. The two intervals $[l/\epsilon - T(\epsilon), l/\epsilon + T(\epsilon)]$ and $[-l - T(\epsilon), -l/\epsilon + T(\epsilon)]$ cut by four brown vertical lines are the places where weighting function comparison occurs in right inverse estimates. For convenience of readers, we also include the schematic picture of the preglued solution u_{app}^ϵ and Floer datum (K_ϵ, J_ϵ) of the perturbed Cauchy-Riemann equation. Note that in interval $\pm[l/\epsilon, \tau(\epsilon)]$, the Hamiltonian $K_\epsilon = 0$ and $J_\epsilon = J_0$ because of the cut-off function $\kappa_\epsilon^\pm(\tau)$.

This Banach manifold can be thought as the gluing of the Banach manifolds \mathcal{B}_\pm and $\mathcal{B}_{\chi_\epsilon}$. More precisely $\mathcal{B}_{res}^\epsilon(z_-, z_+; l/\epsilon)$ ($l \geq l_0 > 0$) consists of maps $u : \Sigma_\epsilon \rightarrow M$ satisfying:

- (1) Σ_ϵ is diffeomorphic $\mathbb{R} \times S^1$ but equipped with the conformal structure induced by the following metric arising from the decomposition into the standard cylinder and hemispheres S_\pm ,

$$\Sigma_\epsilon = S_- \cup ([-\tau(\epsilon), \tau(\epsilon)] \times S^1) \cup S_+. \quad (9.1)$$

- (2) $u \in W_{loc}^{1,p}(\Sigma_\epsilon, M)$
- (3) $\lim_{\tau \rightarrow +\infty} u(\tau, t) = z_+(t)$ and $\lim_{\tau \rightarrow -\infty} u(\tau, t) = z_-(t)$ for all $t \in S^1$.

- (4) For $u \in \mathcal{B}_\varepsilon$, and any variation vector field $\xi \in \Gamma(W^{1,p}(u^*TM))$, we define the Banach norm to be

$$\|\xi\|_\varepsilon = \left\| \tilde{\xi} \right\|_{W_{\beta_{\delta,\varepsilon}}^{1,p}(\Sigma_\varepsilon)} + \|\xi_0\|_{W_\varepsilon^{1,p}(-l/\varepsilon, l/\varepsilon)} + |\xi_0(\pm l/\varepsilon)|, \quad (9.2)$$

where

$$\begin{aligned} \xi_0(\tau) &= \begin{cases} \int_{S^1} \xi(\tau, t) dt, & |\tau| \leq l/\varepsilon \\ \kappa_\varepsilon^0(\tau) \int_{S^1} \xi(\pm l/\varepsilon, t) dt, & |\tau| \geq l/\varepsilon \end{cases}, \\ \tilde{\xi}(\tau, t) &= \xi(\tau, t) - \xi_0(\tau), \\ \left\| \tilde{\xi} \right\|_{W_{\beta_{\delta,\varepsilon}}^{1,p}(\Sigma_\varepsilon)}^p &= \int \int_{\Sigma_\varepsilon} \left(|\tilde{\xi}|^p + |\nabla \tilde{\xi}|^p \right) \beta_{\delta,\varepsilon}(\tau) dvol_{\Sigma_\varepsilon}. \end{aligned}$$

Therefore, we have an ε -family of Banach manifolds $\mathcal{B}_{res}^\varepsilon(z_-, z_+; l/\varepsilon)$, and an ε -family of equations $\bar{\partial}_{(J_\varepsilon, K_\varepsilon)} u^\varepsilon = 0$ defined on each Banach bundle

$$\pi : \mathcal{L}_{res}^\varepsilon(z_-, z_+; l/\varepsilon) \rightarrow \mathcal{B}_{res}^\varepsilon(z_-, z_+; l/\varepsilon),$$

where

$$\mathcal{L}_{res}^\varepsilon(z_-, z_+; l/\varepsilon) = \bigcup_{u \in \mathcal{B}_{res}^\varepsilon(z_-, z_+; l/\varepsilon)} L_{\beta_{\delta,\varepsilon}}^p(\Lambda^{0,1}(u^*TM)).$$

Here each fiber $L_{\beta_{\delta,\varepsilon}}^p(\Lambda^{0,1}(u^*TM))$ consists of sections $\eta \in L^p(\Lambda^{0,1}(u^*TM))$ with $\|\eta\|_\varepsilon < \infty$ and the norm $\|\eta\|_\varepsilon$ is given by

$$\|\eta\|_\varepsilon = \|\tilde{\eta}\|_{L_{\beta_{\delta,\varepsilon}}^p(\Sigma_\varepsilon)} + \|\kappa_\varepsilon^0(\tau) \eta_0(\tau)\|_{L_\varepsilon^p[-l/\varepsilon, l/\varepsilon]},$$

where $\tilde{\eta}, \eta_0$ and $\tilde{\eta}$ are defined similarly as those for ξ , namely

$$\begin{aligned} \eta_0(\tau) &= \begin{cases} \int_{S^1} \eta(\tau, t) dt, & |\tau| < l/\varepsilon \\ 0, & |\tau| \geq l/\varepsilon \end{cases}, \\ \tilde{\eta}(\tau, t) &= \eta(\tau, t) - \eta_0(\tau), \\ \|\tilde{\eta}\|_{L_{\beta_{\delta,\varepsilon}}^p(\Sigma_\varepsilon)}^p &= \int \int_{\Sigma_\varepsilon} |\tilde{\eta}|^p \beta_{\delta,\varepsilon}(\tau) dvol_{\Sigma_\varepsilon}. \end{aligned}$$

We define

$$\mathcal{B}_{res}^\varepsilon(z_-, z_+) = \bigcup_{l \geq l_0} \mathcal{B}_{res}^\varepsilon(z_-, z_+; l/\varepsilon) \quad (9.3)$$

$$\mathcal{L}_{res}^\varepsilon(z_-, z_+) = \bigcup_{l \geq l_0} \mathcal{L}_{res}^\varepsilon(z_-, z_+; l/\varepsilon). \quad (9.4)$$

For $(u, l/\varepsilon) \in \mathcal{B}_{res}^\varepsilon(z_-, z_+)$, its tangent space consists of elements (ξ, μ) where $\xi \in T_u \mathcal{B}_{res}^\varepsilon(z_-, z_+; l/\varepsilon)$ and $\mu \in T_{l/\varepsilon} \mathbb{R}_+ \cong \mathbb{R}$ with the norm

$$\|(\xi, \mu)\|_\varepsilon = \|\xi\|_\varepsilon + |\mu|.$$

Geometrically μ corresponds to the variation of conformal structure of the neck cylinder $[-l/\varepsilon, l/\varepsilon] \times S^1$ by varying the length of the cylinder but keeping the radius of S^1 fixed. For $\mu \in T_{l/\varepsilon} \mathbb{R}$, the induced path in $\mathcal{B}_{res}^\varepsilon(z_-, z_+)$, starting from $u \in \mathcal{B}_{res}^\varepsilon(z_-, z_+; l/\varepsilon)$, is $u_s \in \mathcal{B}_{res}^\varepsilon(z_-, z_+; l/\varepsilon - s\mu)$, where $u_s(\tau', t)$ is the reparameterization of $u(\tau, t)$ on the neck part,

$$u_s(\tau', t) := u\left(\frac{\tau'}{l - s\varepsilon\mu}, t\right), \quad (\tau', t) \in [-(l/\varepsilon - s\mu), l/\varepsilon - s\mu] \times S^1, \quad (9.5)$$

for s nearby 0. There is a canonical way to associate points on u to points on u_s :

$$u(\tau, t) \longleftrightarrow u_s(\tau', t), \quad \text{where } \tau' = \frac{(l - s\varepsilon\mu)\tau}{l}.$$

Using this identification we can realize the variation of conformal structure as a vector field on $[-l/\varepsilon, l/\varepsilon] \times S^1$.

Here we define the Banach norm on $T_u \mathcal{B}_{res}^\varepsilon(z_-, z_+; l/\varepsilon)$ by

$$\|\xi\|_\varepsilon = \|\tilde{\xi}\|_{W_{\beta_{\delta,\varepsilon}}^{1,p}(\mathbb{R} \times S^1)} + \|\xi_0\|_{W_\varepsilon^{1,p}([-l/\varepsilon, l/\varepsilon])} + |\xi_0(\pm l/\varepsilon)|, \quad (9.6)$$

where

$$\begin{aligned} \xi_0(\tau) &= \begin{cases} \int_{S^1} \xi(\tau, t) dt, & |\tau| \leq l/\varepsilon \\ \kappa_\varepsilon^0(\tau) \int_{S^1} \xi(\pm l/\varepsilon, t) dt, & |\tau| \geq l/\varepsilon \end{cases} \\ \tilde{\xi}(\tau, t) &= \xi(\tau, t) - \xi_0(\tau), \\ \|\tilde{\xi}\|_{W_{\beta_{\delta,\varepsilon}}^{1,p}(\mathbb{R} \times S^1)}^p &= \int \int_{\mathbb{R} \times S^1} (|\tilde{\xi}|^p + |\nabla \tilde{\xi}|^p) \beta_{\delta,\varepsilon}(\tau) d\tau dt \end{aligned}$$

This Banach manifold can be thought as the gluing of the Banach manifolds \mathcal{B}_\pm and $\mathcal{B}_{\chi_\varepsilon}$. Similarly for $\eta \in \Gamma(L^p(u^*TM \otimes \Lambda^{0,1}(\Sigma_\varepsilon)))$, the Banach norm is

$$\|\eta\|_\varepsilon = \|\tilde{\eta}\|_{L_{\beta_{\delta,\varepsilon}}^p(\mathbb{R} \times S^1)} + \|\kappa_\varepsilon^0(\tau) \eta_0(\tau)\|_{L_\varepsilon^p([-l/\varepsilon, l/\varepsilon])},$$

where $\tilde{\eta}, \eta_0$ and $\tilde{\eta}$ are defined similarly as those for ξ , namely

$$\begin{aligned} \eta_0(\tau) &= \begin{cases} \int_{S^1} \eta(\tau, t) dt, & |\tau| < l/\varepsilon \\ 0, & |\tau| \geq l/\varepsilon \end{cases} \\ \tilde{\eta}(\tau, t) &= \eta(\tau, t) - \eta_0(\tau), \\ \|\tilde{\eta}\|_{L_{\beta_{\delta,\varepsilon}}^p(\mathbb{R} \times S^1)}^p &= \int \int_{\mathbb{R} \times S^1} |\tilde{\eta}|^p \beta_{\delta,\varepsilon}(\tau) d\tau dt. \end{aligned}$$

Remark 9.1. In the norm $\|\xi\|_\varepsilon$ we could have dropped the term $|\xi_0(\pm l/\varepsilon)|$ but still get an equivalent norm, because from Sobolev embedding we have

$$|\xi_0(\pm l/\varepsilon)| \leq C \|\xi_0\|_{W_\varepsilon^{1,p}([-l/\varepsilon, l/\varepsilon])}$$

where C is uniform for all ε and $l \geq l_0 > 0$. However we still keep the $|\xi_0(\pm l/\varepsilon)|$ term because $\|\tilde{\xi}\|_{W_{\beta_{\delta,\varepsilon}}^{1,p}(\{|\tau| > l/\varepsilon\} \times S^1)} + |\xi_0(\pm l/\varepsilon)|$ mimics the Banach norm of ξ_\pm and $\|\tilde{\xi}\|_{W_{\beta_{\delta,\varepsilon}}^{1,p}(\{|\tau| \leq l/\varepsilon\} \times S^1)} + \|\xi_0(\tau)\|_{W_\varepsilon^{1,p}([-l/\varepsilon, l/\varepsilon])}$ mimics the Banach norm of ξ_{χ_ε} and in this sense the norm $\|\xi\|_\varepsilon$ is roughly the sum of them.

Finally we consider the Fredholm section

$$\mathcal{B}_{res}^\varepsilon(z_-, z_+) \rightarrow \mathcal{L}_{res}^\varepsilon(z_-, z_+)$$

given by

$$(u, l/\varepsilon) \mapsto \bar{\partial}_{K_\varepsilon, J_\varepsilon; l/\varepsilon}(u) : \mathcal{B}_{res}^\varepsilon(z_-, z_+) \rightarrow \mathcal{L}_{res, u}^\varepsilon(z_-, z_+; l/\varepsilon)$$

defined fiberwise over $l/\varepsilon \in \mathbb{R}$. We denote the linearization of this map by D_{para}^ε which has expression

$$D_{para}^\varepsilon(\xi, \mu) = D^\varepsilon(\xi) + \frac{\mu}{l} \nabla \varepsilon f(\chi_\varepsilon). \quad (9.7)$$

10. $\bar{\partial}_{(K_\varepsilon, J_\varepsilon)}$ -ERROR ESTIMATE

In this section, we estimate the norm $\|\bar{\partial}_{(K_\varepsilon, J_\varepsilon)} u_{app}^\varepsilon\|_\varepsilon$. Let $\eta = \bar{\partial}_{(K_\varepsilon, J_\varepsilon)} u_{app}^\varepsilon$. We do this estimation separately in several regions:

(1). When $|\tau| \leq l/\varepsilon$, $u_{app}^\varepsilon(\tau, t) = \chi(\varepsilon\tau)$, so

$$\bar{\partial}_{(K_\varepsilon, J_\varepsilon)} u_{app}^\varepsilon(\tau, t) = \bar{\partial}_{(J_0, \varepsilon f)} \chi(\varepsilon\tau) = \frac{\partial}{\partial \tau} \chi(\varepsilon\tau) - \varepsilon \nabla f(\chi(\varepsilon\tau)) = 0;$$

(2). When $l/\varepsilon \leq |\tau| \leq l/\varepsilon + 1$, say $\tau > 0$ case we have $u_{app}^\varepsilon(\tau, t) = \kappa_\varepsilon^0(\tau) \chi(\varepsilon\tau) + (1 - \kappa_\varepsilon^0(\tau)) u_\pm^\varepsilon(\tau, t)$, so

$$\begin{aligned} |\eta(\tau, t)| &= |\bar{\partial}_{(J_0, \varepsilon f)} u_{app}^\varepsilon| \\ &= \left| (\kappa_\varepsilon^0(\tau))' (\chi(\varepsilon\tau) - u_\pm^\varepsilon) + (1 - \kappa_\varepsilon^0(\tau)) \bar{\partial}_{(J_0, \varepsilon f)} u_\pm^\varepsilon \right| \\ &\leq C (|\chi(\varepsilon\tau) - u_\pm^\varepsilon(\tau, t)| + |\varepsilon \nabla f(u_\pm^\varepsilon(\tau, t))|) \\ &\leq \tilde{C}(l) \varepsilon + C\varepsilon := C(l) \varepsilon, \end{aligned} \quad (10.1)$$

where in the first inequality we have used $\bar{\partial}_{(J_0, \varepsilon f)} u_\pm^\varepsilon = \varepsilon \nabla f(u_\pm^\varepsilon)$ and in the second inequality used (8.6) and (8.7), and $\tilde{C}(l) = Cl^{-\frac{p-1}{\delta}}$. The weight $\beta_{\delta, \varepsilon}$ in this interval satisfies

$$\beta_{\delta, \varepsilon}(\tau) = e^{2\pi\delta(-\tau + \tau(\varepsilon))} \leq e^{2\pi\delta(-l/\varepsilon + \tau(\varepsilon))} = e^{(p-1)\ln(1+l/\varepsilon)} \leq D(l) \varepsilon^{1-p},$$

where the constant $D(l) \approx l^{p-1}$. Since $|\tau| \geq l/\varepsilon$, we do not need to distinguish the 0-mode and higher mode of η . So we have

$$\|\eta(\tau)\|_{L_{\beta_{\delta, \varepsilon}}^p([l/\varepsilon, l/\varepsilon+1] \times S^1)} \leq \left(\int_{l/\varepsilon}^{l/\varepsilon+1} (C(l) \varepsilon)^p \cdot D(l) \varepsilon^{1-p} d\tau \right)^{\frac{1}{p}} := E(l) \varepsilon^{\frac{1}{p}},$$

Similarly result holds for $-l/\varepsilon - 1 \leq \tau \leq -l/\varepsilon$ case.

(3). When $|\tau| > l/\varepsilon$, say $\tau > l/\varepsilon$, we recall

$$u_{app}^\varepsilon(\tau, t) = u_+ \circ I_{\tau(\varepsilon)}(\tau, t) = u_+(\tau - \tau(\varepsilon), t)$$

and so satisfies

$$\bar{\partial}_{(K_\varepsilon, J_\varepsilon)} u_{app}^\varepsilon(\tau, t) = \bar{\partial}_{(K_\varepsilon^+, J_\varepsilon^+)} u_+(\tau - \tau(\varepsilon), t) = 0.$$

Similarly $\bar{\partial}_{(K_\varepsilon, J_\varepsilon)} u_{app}^\varepsilon(\tau, t) = 0$ for $\tau < -l/\varepsilon$.

Combining the 3 pieces, we have

$$\|\bar{\partial}_{(J, \varepsilon f)} u_{app}^\varepsilon\|_{L_{\beta_{\delta, \varepsilon}}^p(\mathbb{R} \times S^1)} \leq E(l) \varepsilon^{\frac{1}{p}}, \quad (10.2)$$

where the constant $E(l) \approx \left(C + l^{-\frac{p-1}{\delta}}\right) l^{\frac{p-1}{p}}$.

Remark 10.1. If we use $u_{app}^\varepsilon = \exp_{\chi(\varepsilon\tau)} \left[(1 - \kappa_\varepsilon^0(\tau)) \exp_{\chi(\varepsilon\tau)}^{-1} (u_\pm^\varepsilon(\tau, t)) \right]$ for $\tau \in$

$[l/\varepsilon, l/\varepsilon + 1]$, then $|\eta| = |\bar{\partial}_{(J_0, \varepsilon f)} u_{app}^\varepsilon|$ is controlled pointwise by $C\varepsilon$ as above case (2), plus the C^1 difference between $\exp_{\chi(\varepsilon\tau)} \left[(1 - \kappa_\varepsilon^0(\tau)) \exp_{\chi(\varepsilon\tau)}^{-1} (u_\pm^\varepsilon(\tau, t)) \right]$ and

$\kappa_\varepsilon^0(\tau) \chi(\varepsilon\tau) + (1 - \kappa_\varepsilon^0(\tau)) u_-^\varepsilon(\tau, t)$, which is also of order $C\varepsilon$. Therefore we get the same pointwise estimate

$$|\eta| = |\bar{\partial}_{(J_0, \varepsilon f)} u_{app}^\varepsilon| \leq C\varepsilon.$$

Continuing the remaining steps in case (2) we get the same $L_{\beta_{\delta, \varepsilon}}^p$ estimate

$$\|\eta\|_{L_{\beta_{\delta, \varepsilon}}^p(\pm[l/\varepsilon - 1, l/\varepsilon] \times S^1)} \leq E(l) \varepsilon^{\frac{1}{p}}.$$

The case (1) and (3) are the same as above. Therefore we still get

$$\|\bar{\partial}_{(J_0, \varepsilon f)} u_{app}^\varepsilon\|_{L_{\beta_{\delta, \varepsilon}}^p(\mathbb{R} \times S^1)} \leq E(l) \varepsilon^{\frac{1}{p}}.$$

Remark 10.2. The constant $E(l)$ is bounded if l is in a bounded interval, because we see from the above estimates

$$E(l) \approx \left(C + l^{-\frac{p-1}{\delta}}\right) l^{\frac{p-1}{p}}.$$

So if we assume that $l_0 \leq l \leq L$, then we get uniform $\bar{\partial}_{(K_\varepsilon, J_\varepsilon)}$ error estimate. When $l \rightarrow \infty$, $E(l) \rightarrow \infty$, so the above estimate in (10.1) is too coarse. But notice that when $l \rightarrow \infty$, $\nabla f(\chi(\pm l))$ has exponential decay e^{-cl} , so using this in (10.1) one can get even better $\bar{\partial}_{(K_\varepsilon, J_\varepsilon)}$ error estimate with $E(l) \approx \left(Ce^{-cl} + l^{-\frac{p-1}{\delta}}\right) l^{\frac{p-1}{p}} \rightarrow 0$ as $l \rightarrow \infty$. Thus the error estimate is uniform for all $l \geq l_0$.

11. THE COMBINED RIGHT INVERSE

To keep notation simple, in the following we denote $D^\varepsilon := D_{u_{app}^\varepsilon} \bar{\partial}_{(K_\varepsilon, J_\varepsilon)}$ for the linearization of

$$\bar{\partial}_{(K_\varepsilon, J_\varepsilon)} : T_{(u, l/\varepsilon)} \mathcal{B}_{res}^\varepsilon(z_-, z_+; l/\varepsilon) \rightarrow \mathcal{L}_{res; (u, l/\varepsilon)}^\varepsilon(z_-, z_+; l).$$

In this section, we first construct an approximate right inverse, denoted by $Q_{para}^{app; \varepsilon}$, of the parameterized $\bar{\partial}_{(K_\varepsilon, J_\varepsilon)}$ which allows l to vary, which we denote by

$$D_{para}^\varepsilon : T_{(u, l/\varepsilon)} \mathcal{B}_{res}^\varepsilon(z_-, z_+) \rightarrow \mathcal{L}_{res; (u, l/\varepsilon)}^\varepsilon(z_-, z_+)$$

(see (9.7)) by gluing the right inverses Q_\pm of $D\bar{\partial}_{(K_\pm, J_\pm)}(u_\pm)$ and another operator that takes care of the part of the gradient segment in the middle. We recall

$$T_{(u, l/\varepsilon)} \mathcal{B}_{res}^\varepsilon(z_-, z_+) = T_u \mathcal{B}_{res}^\varepsilon(z_-, z_+; l/\varepsilon) \oplus T_{l/\varepsilon} \mathbb{R} = W_{\rho_\varepsilon}^{(1, p)}(u^* TM) \oplus \mathbb{R}$$

and

$$\mathcal{L}_{res; (u, l/\varepsilon)}^\varepsilon(z_-, z_+) = \mathcal{L}_{res; u}^\varepsilon(z_-, z_+; l/\varepsilon) = L_{\rho_\varepsilon}^p(u^* TM).$$

Therefore the image of $Q_{para}^{app; \varepsilon}$ at $(u, l/\varepsilon)$ is decomposed into

$$Q_{para}^{app; \varepsilon}(\eta) = (\xi_\varepsilon, \mu), \quad \xi_\varepsilon \in W_{\rho_\varepsilon}^{(1, p)}(u^* TM), \quad \mu \in \mathbb{R}.$$

We will define each of ξ_ε and μ in describing the image of the operator $Q_{para}^{app; \varepsilon}(\eta)$.

We introduce several cut-off functions:

- (1) $\kappa_0^\varepsilon = \kappa_0^\varepsilon(\tau)$ is the characteristic function of the interval $[-l/\varepsilon, l/\varepsilon] \subset \mathbb{R}$,
- (2) φ_0^K is the cut-off function defined by

$$\varphi_0^K(\tau) = \begin{cases} 1 & \text{for } |\tau| \leq K \\ 0 & \text{for } |\tau| > K + 1, \end{cases}$$

(3) φ_+^K is the cut-off function

$$\varphi_+^K(\tau) = \begin{cases} 1 & \text{for } \tau \leq K \\ 0 & \text{for } \tau > K+1, \end{cases}$$

and φ_-^K is the function defined by $\varphi_-^K(\tau) = 1 - \varphi_+^K(\tau)$.

Now let $\eta \in \mathcal{L}_{res}^\varepsilon(z_-, z_+)$. We split η into 3 pieces of \mathbb{R} according to the division of the expression of the approximate solution (8.4):

$$\eta|_{(-\infty, l/\varepsilon]}, \eta|_{[l/\varepsilon, \infty)}, \eta|_{[-l/\varepsilon, l/\varepsilon]}. \quad (11.1)$$

Multiplying the characteristic functions

$$\kappa_-^\varepsilon, \kappa_+^\varepsilon, \kappa_0^\varepsilon$$

of the corresponding intervals, we regard each of them defined on the whole cylinder $\mathbb{R} \times S^1$. (Remark: the smooth cut-off functions we used before are $\kappa_-^\varepsilon, \kappa_+^\varepsilon, \kappa_0^\varepsilon$, whose notations are similar to characteristic functions but lower and upper indices are switched.)

Consider the translations of the first and the third pieces

$$\eta_\pm^\varepsilon(\tau, t) := (\kappa_\pm^\varepsilon \eta) \circ I_{\pm\tau(\varepsilon)}(\tau, t) = (\kappa_\pm^\varepsilon \eta)(\tau - \pm\tau(\varepsilon), t)$$

whose supports become

$$\left(-\infty, \frac{p-1}{\delta}S(\varepsilon)\right) \times S^1, \left(-\frac{p-1}{\delta}S(\varepsilon), \infty\right) \times S^1$$

respectively. Then we define

$$\xi_\pm^\varepsilon = Q_\pm(\eta_\pm^\varepsilon) \circ I_{\pm(-\tau(\varepsilon))}. \quad (11.2)$$

Now we consider the middle piece $\kappa_0^\varepsilon \eta$ which is supported on $[-l/\varepsilon, l/\varepsilon] \times S^1$. To describe this piece precisely, we need some careful examination how the Banach manifold (9.3) and the tangent vectors thereof at the approximate solution u_ε near (u_-, χ, u_+, l) are made of, and how the operator $\bar{\partial}_{(K_\varepsilon, J_\varepsilon)}$ acts on $T_{u_\varepsilon} \mathcal{B}_{res}^\varepsilon(z_-, z_+)$.

Recall χ satisfies $\dot{\chi} + \text{grad } f(\chi) = 0$. Since we assume that the pair (f, g) is Morse-Smale, the linearization

$$D_\chi = \nabla_\tau + \nabla \text{grad } f(\chi)$$

is invertible as mentioned before. We denote by Q_χ its right inverse.

We denote the renormalized $\chi : [-l, l] \rightarrow M$ by

$$\chi_\varepsilon : [-l/\varepsilon, l/\varepsilon] \rightarrow M, \quad \chi_\varepsilon(\tau) := \chi(\varepsilon\tau)$$

which satisfies $\dot{\chi}_\varepsilon + \varepsilon \text{grad } f(\chi) = 0$. We apply the right inverse $Q_{\chi_\varepsilon}^{para}$ (allowing l to vary) of $D_{\chi_\varepsilon}^{para}$ to $\kappa_0^\varepsilon \eta$ and write it as

$$Q_{\chi_\varepsilon}^{para}(\kappa_0^\varepsilon \eta) = (\xi_{\chi_\varepsilon}, \mu).$$

We recall the operator

$$D_{para}^\varepsilon : W^{1,p}(\chi_\varepsilon^* TM \times S^1) \oplus \mathbb{R} \rightarrow L^p(\chi_\varepsilon^* TM \times S^1)$$

has the form

$$D_{para}^\varepsilon(\xi, \mu) = D_{\chi_\varepsilon}(\xi) + \frac{\mu}{l} \nabla_\varepsilon f(\chi_\varepsilon)$$

with $D_{\chi_\varepsilon} = \frac{\partial}{\partial \tau} + J_0 \frac{\partial}{\partial t} + \nabla \text{grad}(\varepsilon f)$. First we define

$$\xi_{\chi_\varepsilon} = Q_{\chi_\varepsilon}(\eta_{\chi_\varepsilon})$$

where Q_{χ_ε} is the right inverse of D_{χ_ε} constructed in section 5. Then we determine μ by solving the 0-mode matching condition

$$\left(\xi_{\chi_\varepsilon}\right)_0 (\pm l/\varepsilon) + \frac{\varepsilon\mu}{l} \nabla f(p_\pm) = \xi_\pm(o_\pm). \quad (11.3)$$

Now we are ready to write down the formula for $(\xi_\varepsilon, \mu) = Q_{para}^{app;\varepsilon}(\eta)$. Here we define

$$\xi_\varepsilon = \begin{cases} \xi_+^\varepsilon, & l/\varepsilon + T(\varepsilon) \leq \tau, \\ Pal_{\chi,\varepsilon} \left[\left(\xi_{\chi_\varepsilon}\right)_0 + \phi_0^{l/\varepsilon+T(\varepsilon)}(\tau) \left(\xi_{\chi_\varepsilon} - \left(\xi_{\chi_\varepsilon}\right)_0\right) \right] \\ \quad + Pal_{+,\varepsilon} \left[\phi_+^{(l/\varepsilon-T(\varepsilon))}(\tau) (\xi_+^\varepsilon - v_+) \right], & l/\varepsilon \leq \tau \leq l/\varepsilon + T(\varepsilon), \\ \xi_{\chi_\varepsilon}, & -l/\varepsilon \leq \tau \leq l/\varepsilon, \\ Pal_{\chi,\varepsilon} \left[\left(\xi_{\chi_\varepsilon}\right)_0 + \phi_0^{l/\varepsilon+T(\varepsilon)}(\tau) \left(\xi_{\chi_\varepsilon} - \left(\xi_{\chi_\varepsilon}\right)_0\right) \right] \\ \quad + Pal_{-,\varepsilon} \left[\phi_-^{(l/\varepsilon-T(\varepsilon))}(\tau) (\xi_-^\varepsilon - v_-) \right], & -l/\varepsilon - T(\varepsilon) \leq \tau \leq -l/\varepsilon, \\ \xi_-^\varepsilon, & \tau \leq -l/\varepsilon - T(\varepsilon), \end{cases}$$

where

$$\begin{aligned} \xi_{\chi_\varepsilon} &= Q_{\chi_\varepsilon} \circ (\kappa_0^\varepsilon \eta), \\ \left(\xi_{\chi_\varepsilon}\right)_0 &= \int_{S^1} \xi_{\chi_\varepsilon} dt, \\ \xi_\pm^\varepsilon &= Q_\pm(\eta_\pm^\varepsilon) \circ I_{\pm(-\tau(\varepsilon))} \end{aligned}$$

and μ is determined by (11.3).

Here we recall from (7.1) that v_\pm is defined as the *cut-off vector field* of the constant vector field extending the vector $\xi_\pm(o_\pm)$,

$$v_\pm := \varphi_\pm(\tau, t) Pal_\pm(\tau, t) (\xi_\pm(o_\pm)),$$

$\varphi_\pm(\tau, t)$ is a cut-off function, $\varphi_\pm(\tau, t) = 1$ near o_\pm and $\varphi_\pm(\tau, t) = 0$ outside the cylindrical neighborhood of o_\pm , $Pal_\pm(\tau, t)$ is the parallel transport along the shortest geodesics from $u_\pm(o_\pm)$ to $u_\pm(\tau, t)$, and the $Pal_{\chi,\varepsilon}$ and $Pal_{\pm,\varepsilon}$ are parallel transports from χ and u_\pm to the corresponding points on u_{app}^ε along the shortest geodesics respectively.

The interpolation happens on the regions

$$\pm[l/\varepsilon - T(\varepsilon), l/\varepsilon - T(\varepsilon) + 1] \text{ and } \pm[l/\varepsilon + T(\varepsilon), l/\varepsilon + T(\varepsilon) + 1],$$

which avoids the peak $\tau = \pm l/\varepsilon$ of the weighting function $\beta_{\delta,\varepsilon}$. Here we choose $T(\varepsilon) > 0$ so that as $\varepsilon \rightarrow 0$, it behaves as

$$T(\varepsilon) \rightarrow \infty, \quad \varepsilon T(\varepsilon) \rightarrow 0, \quad T(\varepsilon) < \frac{p-1}{\delta} S(\varepsilon)$$

For example, we can take and fix

$$T(\varepsilon) = \frac{1}{3} \frac{p-1}{\delta} S(\varepsilon) \quad (11.4)$$

henceforth.

We note that in this construction we have ξ_{χ_ε} defined for all $\tau \in \mathbb{R}$, solving $D_\varepsilon \xi_{\chi_\varepsilon} = \kappa_0^\varepsilon \eta$ which is equivalent to the first order linear ODE

$$\frac{\partial \xi}{\partial \tau} + J_0 \frac{\partial \xi}{\partial t} + \varepsilon \nabla_\xi \text{grad} f = \kappa_0^\varepsilon \eta$$

on $\mathbb{R} \times S^1$, not just on $[-l/\varepsilon, l/\varepsilon]$. Since $\kappa_0^\varepsilon \eta \in L^p$, ξ_{χ_ε} lies in $W^{1,p}$ and in particular is continuous on $\mathbb{R} \times S^1$. Therefore, considering the evaluation of $(\xi_{\chi_\varepsilon})_0$ at $(\pm(l/\varepsilon + T(\varepsilon)))$ makes sense. Thinking ξ_{χ_ε} this way is to avoid its cut-off too close to $\tau = \pm l/\varepsilon$, the peak of the weighting function. Plus, later we will show the $W^{1,p}$ norm of ξ_{χ_ε} on $[-l/\varepsilon, l/\varepsilon] \times S^1$ controls its $W^{1,p}$ norm on the whole $\mathbb{R} \times S^1$.

Remark 11.1. In the construction of $(Q_{para}^\varepsilon)_{actual}$, since the interpolation happens in a $C\varepsilon$ radius ball around p_\pm , we can use the connection from the constant Euclidean metric g_{p_\pm} on the Weinstein neighborhood around p_\pm to identify different tangent spaces such that the vector sum \pm makes sense without parallel transport. We call this approximate right inverse as $Q_{para}^{app;\varepsilon}$, which is simpler in exposition and estimates:

$$\xi_\varepsilon = \begin{cases} \xi_+^\varepsilon, & l/\varepsilon + T(\varepsilon) \leq \tau, \\ (\xi_{\chi_\varepsilon})_0 + \phi_0^{l/\varepsilon+T(\varepsilon)}(\tau) (\xi_{\chi_\varepsilon} - (\xi_{\chi_\varepsilon})_0) \\ \quad + \phi_+^{(l/\varepsilon-T(\varepsilon))}(\tau) (\xi_+^\varepsilon - v_+), & l/\varepsilon \leq \tau \leq l/\varepsilon + T(\varepsilon), \\ \xi_{\chi_\varepsilon}, & -l/\varepsilon \leq \tau \leq l/\varepsilon, \\ (\xi_{\chi_\varepsilon})_0 + \phi_0^{l/\varepsilon+T(\varepsilon)}(\tau) (\xi_{\chi_\varepsilon} - (\xi_{\chi_\varepsilon})_0) \\ \quad + \phi_-^{(l/\varepsilon-T(\varepsilon))}(\tau) (\xi_-^\varepsilon - v_-), & -l/\varepsilon - T(\varepsilon) \leq \tau \leq -l/\varepsilon, \\ \xi_-^\varepsilon, & \tau \leq -l/\varepsilon - T(\varepsilon), \end{cases}$$

The interpolation in $(Q_{para}^{app;\varepsilon})_{actual}$ uses the parallel transport from the nonconstant metric g inside the $C\varepsilon$ radius ball, but the difference is a smooth tensor of order $C\varepsilon$, first in C^1 pointwise then in operator norm, namely

$$(Q_{para}^{app;\varepsilon})_{actual} = Q_{para}^{app;\varepsilon} + H_\varepsilon$$

with the operator norm $\|H_\varepsilon\| \leq C\varepsilon$. This will not effect the approximate inverse. Therefore it is suffice to estimate the above (simpler) $Q_{para}^{app;\varepsilon}$ in the next section.

12. ESTIMATE OF THE COMBINED RIGHT INVERSE

Proposition 12.1. $Q_{para}^{app;\varepsilon}$ has a uniform bound independent on ε . More precisely, there exists a uniform constant C such that for all η ,

$$\|Q_{para}^{app;\varepsilon} \eta\|_\varepsilon \leq C \left(\|\xi_{\chi_\varepsilon}\|_{W_{\rho_\varepsilon}^{1,p}([-l/\varepsilon, l/\varepsilon] \times S^1)} + \|\xi_\pm\|_{W_\alpha^{1,p}(\Sigma_\pm)} \right) \leq C \|\eta\|_\varepsilon.$$

Proof. The proof is by splitting the Banach norm on u_{app}^ε to those associated to u_{χ_ε} and u_\pm . We also need to estimates on

- (1) uniform convergence of $\xi_\pm(\tau)$ and $\xi_{\chi_\varepsilon}(\tau)$ into v_\pm , because our Banach norms involve first taking out the Morse-Bott variation,
- (2) and the matching constant μ

in terms of the norm $\|\eta\|_\varepsilon$. In the interpolation region $Q_{para}^{app;\varepsilon} \eta = (\xi_\varepsilon, \mu)$ is given by

$$\xi_\varepsilon = (\xi_{\chi_\varepsilon})_0 + \phi_0^{l/\varepsilon+T(\varepsilon)}(\tau) (\xi_{\chi_\varepsilon} - (\xi_{\chi_\varepsilon})_0) + \phi_+^{(l/\varepsilon-T(\varepsilon))}(\tau) (\xi_+^\varepsilon - v_+),$$

when $\tau \geq 0$ and μ by solving the matching condition

$$(\xi_{\chi_\varepsilon})_0(l/\varepsilon) = v_+ - \varepsilon \frac{\mu}{l} \nabla f(p_+).$$

We also recall the definition of the Banach norm

$$\|(\xi, \mu)\|_\varepsilon = \|\xi\|_\varepsilon + |\mu| = \left\| \tilde{\xi} \right\|_{W_{\beta_{\delta, \varepsilon}}^{1,p}(\mathbb{R} \times S^1)} + \|\xi_0(\tau)\|_{W_\varepsilon^{1,p}([-l/\varepsilon, l/\varepsilon])} + |\xi_0(\pm l/\varepsilon)| + |\mu|.$$

With these preparation, we are now ready to carry out the estimate of the norm $\|Q_{para}^{app;\varepsilon}(\eta)\|_\varepsilon$. For the μ component, since $\left((\xi_{\chi_\varepsilon})_0, \mu\right) = \left(Q_{\chi_\varepsilon}^{para}\right)_0 \circ \kappa_0^\varepsilon \eta$ and $\left(Q_{\chi_\varepsilon}^{para}\right)_0$ has uniform bound (one can use (5.23) to see the $\left(Q_{\chi_\varepsilon}^{para}\right)_0$ has the same operator bound as $\left(Q_\chi^{para}\right)_0$ for all ε), we have

$$|\mu| \leq C \|(\kappa_0^\varepsilon \eta)_0\|_{L_\varepsilon^p([-l/\varepsilon, l/\varepsilon])} \leq C \|\eta\|_\varepsilon.$$

For the $\|\xi_\varepsilon\|_\varepsilon$ component, we consider two regions separately, one on $[0, l/\varepsilon] \times S^1$ and the other on $[l/\varepsilon, \tau(\varepsilon)] \times S^1 = [l/\varepsilon, l/\varepsilon + \frac{p-1}{\delta}S(\varepsilon)] \times S^1$.

On $[0, l/\varepsilon] \times S^1$. Since $\phi_0^{l/\varepsilon+T(\varepsilon)}(\tau) \equiv 1$, we have

$$\begin{aligned} \xi_\varepsilon &= \left(\xi_{\chi_\varepsilon}\right)_0 + \left(\xi_{\chi_\varepsilon} - \left(\xi_{\chi_\varepsilon}\right)_0\right) + \phi_+^{l/\varepsilon-T(\varepsilon)}(\tau) (\xi_+^\varepsilon - v_+) \\ &= \xi_{\chi_\varepsilon} + \phi_+^{(l/\varepsilon-T(\varepsilon))}(\tau) (\xi_+^\varepsilon - v_+). \end{aligned}$$

Therefore for the 0-mode $(\xi_\varepsilon)_0$ we have

$$\begin{aligned} \|(\xi_\varepsilon)_0\|_{W_\varepsilon^{1,p}[0, l/\varepsilon]} &= \left\| \left(\xi_{\chi_\varepsilon}\right)_0 + \phi_+^{l/\varepsilon-T(\varepsilon)}(\tau) (\xi_+^\varepsilon - v_+)_0 \right\|_{W_\varepsilon^{1,p}([0, l/\varepsilon])} \\ &\leq \left\| \left(\xi_{\chi_\varepsilon}\right)_0 \right\|_{W_\varepsilon^{1,p}([0, l/\varepsilon])} + \left\| \phi_+^{l/\varepsilon-T(\varepsilon)}(\tau) (\xi_+^\varepsilon - v_+)_0 \right\|_{W_\varepsilon^{1,p}([0, l/\varepsilon])} \\ &\leq \left\| \xi_{\chi_\varepsilon} \right\|_{W_{\rho_\varepsilon}^{1,p}(\mathbb{R} \times S^1)} + C \left\| (\xi_+^\varepsilon - v_+)_0 \cdot \varepsilon^{\frac{1-p}{p}} \right\|_{W^{1,p}([0, l/\varepsilon] \times S^1)} \\ &\leq \left\| \xi_{\chi_\varepsilon} \right\|_{W_{\rho_\varepsilon}^{1,p}(\mathbb{R} \times S^1)} + C \|\xi_+\|_{W_\alpha^{1,p}([0, l/\varepsilon] \times S^1)}, \end{aligned}$$

where the third inequality is because $\|\cdot\|_{W_\varepsilon^{1,p}}$ is a component of $\|\cdot\|_{W_{\rho_\varepsilon}^{1,p}}$, and the last inequality holds because for $\tau \in [0, l/\varepsilon]$, the exponential weight

$$e^{2\pi\delta(-\tau+\tau(\varepsilon))} = e^{2\pi\delta(-\tau+\tau(\varepsilon))} \geq e^{2\pi\delta(\frac{p-1}{\delta}S(\varepsilon))} = (1 + l/\varepsilon)^{p-1} \geq C\varepsilon^{1-p},$$

then

$$\begin{aligned} &\left\| (\xi_+^\varepsilon - v_+)_0 \cdot \varepsilon^{\frac{1-p}{p}} \right\|_{W^{1,p}([0, l/\varepsilon] \times S^1)} \\ &\leq C \left\| (\xi_+^\varepsilon - v_+)_0 \cdot e^{\frac{2\pi\delta}{p}(-\tau+\tau(\varepsilon))} \right\|_{W^{1,p}([0, l/\varepsilon] \times S^1)} \\ &= C \left\| (\xi_+^\varepsilon - v_+)_0 \cdot e^{\frac{2\pi\delta}{p}|\tau|} \right\|_{W^{1,p}([- \tau(\varepsilon), -\frac{p-1}{\delta}S(\varepsilon)] \times S^1)} \\ &\leq C \|\xi_+\|_{W_\alpha^{1,p}(\Sigma_+)}. \end{aligned}$$

For the higher mode $\tilde{\xi}_\varepsilon$, we have

$$\begin{aligned} \|\tilde{\xi}_\varepsilon\|_{W_{\rho_\varepsilon}^{1,p}([0, l/\varepsilon] \times S^1)} &= \left\| \widetilde{\xi_{\chi_\varepsilon}} + \phi_+^{l/\varepsilon-T(\varepsilon)}(\tau) \left(\widetilde{(\xi_+^\varepsilon - v_+)} \right) \right\|_{W_{\rho_\varepsilon}^{1,p}([0, l/\varepsilon] \times S^1)} \\ &\leq \left\| \widetilde{\xi_{\chi_\varepsilon}} \right\|_{W_{\rho_\varepsilon}^{1,p}([0, l/\varepsilon] \times S^1)} + C \left\| \widetilde{(\xi_+^\varepsilon - v_+)} \right\|_{W_{\rho_\varepsilon}^{1,p}([0, l/\varepsilon] \times S^1)} \\ &\leq C \left(\left\| \xi_{\chi_\varepsilon} \right\|_{W_{\rho_\varepsilon}^{1,p}(\mathbb{R} \times S^1)} + \|\xi_+\|_{W_\alpha^{1,p}(\Sigma_+)} \right). \end{aligned}$$

The last inequality holds because the exponential weight $e^{2\pi\delta(-\tau+\tau(\varepsilon))}$ of $W_\alpha^{1,p}$ is bigger than the power weight $\varepsilon^{1-p+\delta} (1+|\tau|)^\delta$ of $W_{\rho_\varepsilon}^{1,p}$ on $[0, l/\varepsilon] \times S^1$, by

$$e^{2\pi\delta(-\tau+\tau(\varepsilon))} \geq (1+l/\varepsilon)^{p-1} = C\varepsilon^{1-p+\delta} (1+l/\varepsilon)^\delta \geq C\varepsilon^{1-p+\delta} (1+|\tau|)^\delta.$$

In the last inequality we also used that the projection operators

$$P_0 : \xi \rightarrow \xi_0 \text{ and } \tilde{P} : \xi \rightarrow \tilde{\xi}$$

from $W_\delta^{1,p}$ to $W_\delta^{1,p}$ are uniformly bounded operators by Hölder inequality on S^1

$$\begin{aligned} \int_I (|\xi_0|^p + |\nabla_\tau \xi_0|^p) e^{2\pi\delta|\tau|} d\tau &= \int_I \left(\left| \int_{S^1} \xi dt \right|^p + \left| \int_{S^1} \nabla_\tau \xi dt \right|^p \right) e^{2\pi\delta|\tau|} d\tau \\ &\leq \int_I \left(\int_{S^1} |\xi|^p dt + \int_{S^1} |\nabla \xi|^p dt \right) e^{2\pi\delta|\tau|} d\tau \end{aligned}$$

for any interval I .

On $[l/\varepsilon, \tau(\varepsilon)] \times S^1$. On this region the contribution to $\|\xi_\varepsilon\|_\varepsilon$ is

$$|(\xi_\varepsilon)_0(l/\varepsilon)| + \|\xi_\varepsilon(\tau, t) - (\xi_\varepsilon)_0(l/\varepsilon)\|_{W_{\beta_{\delta,\varepsilon}}^{1,p}([l/\varepsilon, \tau(\varepsilon)] \times S^1)}.$$

From the matching condition (6.13), we have

$$(\xi_\varepsilon)_0(l/\varepsilon) = \left(\xi_{\chi_\varepsilon} \right)_0(l/\varepsilon) + (\xi_+)_0(l/\varepsilon) - v_+ = (\xi_+)_0(l/\varepsilon) - \varepsilon \frac{\mu}{l} \nabla f(p_+).$$

Therefore for the term $(\xi_\varepsilon)_0(l/\varepsilon)$,

$$\begin{aligned} |(\xi_\varepsilon)_0(l/\varepsilon)| &\leq |(\xi_+)_0(l/\varepsilon)| + \left| \frac{\varepsilon}{l} \nabla f(p_+) \right| |\mu| \\ &\leq C \|\xi_+\|_{W_\alpha^{1,p}(\Sigma_+)} + C\varepsilon |\mu| \\ &\leq C \|\eta_+\|_{L_\alpha^p(\Sigma_+)} + C\varepsilon \|\eta_\varepsilon\|_\varepsilon \leq C \|\eta\|_\varepsilon. \end{aligned}$$

For $\tilde{\xi}_\varepsilon = \xi_\varepsilon(\tau, t) - (\xi_\varepsilon)_0(l/\varepsilon)$,

$$\begin{aligned} \xi_\varepsilon(\tau, t) - (\xi_\varepsilon)_0(l/\varepsilon) &= \left(\xi_{\chi_\varepsilon} \right)_0(\tau) + \phi_0^{l/\varepsilon+T(\varepsilon)}(\tau) \left(\xi_{\chi_\varepsilon} - \left(\xi_{\chi_\varepsilon} \right)_0 \right) \\ &\quad + (\xi_+^\varepsilon(\tau, t) - v_+) - (\xi_+)_0(l/\varepsilon) + \varepsilon \frac{\mu}{l} \nabla f(p_+) \\ &= [\xi_+^\varepsilon(\tau, t) - (\xi_+)_0(l/\varepsilon)] + \phi_0^{l/\varepsilon+T(\varepsilon)}(\tau) \left(\xi_{\chi_\varepsilon} - \left(\xi_{\chi_\varepsilon} \right)_0 \right) \\ &\quad + \left[\left(\xi_{\chi_\varepsilon} \right)_0(\tau) - \left(\xi_{\chi_\varepsilon} \right)_0(l/\varepsilon) \right], \end{aligned} \tag{12.1}$$

where in the last row we have used (6.13). For the first term, same as the computation (10.55) \sim (10.60) in [OZ1] Lemma 10.10, we have

$$\begin{aligned} \int_{S^1} \int_{l/\varepsilon}^{\tau(\varepsilon)} |\xi_+^\varepsilon(\tau, t) - (\xi_+)_0(l/\varepsilon)|^p \\ + |\nabla (\xi_+^\varepsilon(\tau, t) - (\xi_+)_0(l/\varepsilon))|^p e^{2\pi\delta(-\tau+\tau(\varepsilon))} d\tau dt \leq C \|\xi_+\|_{W_\alpha^{1,p}(\Sigma_+)}^p \end{aligned} \tag{12.2}$$

For the second term, by comparing the exponential weight and power weight on $[l/\varepsilon, l/\varepsilon + \frac{p-1}{\delta} S(\varepsilon)]$

$$e^{2\pi\delta(-\tau+\tau(\varepsilon))} = (1+l/\varepsilon)^{p-1} e^{-2\pi(\tau-l/\varepsilon)} \leq C\varepsilon^{1-p} (1+|\tau|)^\delta, \tag{12.3}$$

we have

$$\begin{aligned}
& \left\| \phi_0^{l/\varepsilon+T(\varepsilon)}(\tau) \left(\xi_{\chi_\varepsilon} - \left(\xi_{\chi_\varepsilon} \right)_0 \right) e^{\frac{2\pi\delta}{p}(-\tau+\tau(\varepsilon))} \right\|_{W^{1,p}([l/\varepsilon, \tau(\varepsilon)] \times S^1)} \\
& \leq C \left\| \phi_0^{l/\varepsilon+T(\varepsilon)}(\tau) \left(\xi_{\chi_\varepsilon} - \left(\xi_{\chi_\varepsilon} \right)_0 \right) \left(\varepsilon^{1-p} (1+|\tau|)^\delta \right)^{\frac{1}{p}} \right\|_{W^{1,p}([l/\varepsilon, \tau(\varepsilon)] \times S^1)} \\
& = C \left\| \xi_{\chi_\varepsilon} - \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{W_{\rho_\varepsilon}^{1,p}(\mathbb{R} \times S^1)} \\
& \leq C \left\| \xi_{\chi_\varepsilon} \right\|_{W_{\rho_\varepsilon}^{1,p}(\mathbb{R} \times S^1)}. \tag{12.4}
\end{aligned}$$

For the third term, note that from (5.25)

$$\left| \left(\xi_{\chi_\varepsilon} \right)_0(\tau) - \left(\xi_{\chi_\varepsilon} \right)_0(l/\varepsilon) \right| \leq |\varepsilon\tau \pm l|^\gamma \left\| \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{W_\varepsilon^{1,p}([l/\varepsilon, \tau(\varepsilon)])}$$

where $\gamma = 1 - \frac{1}{p}$, we have

$$\begin{aligned}
& \int_{S^1} \int_{l/\varepsilon}^{\tau(\varepsilon)} \left| \left(\xi_{\chi_\varepsilon} \right)_0(\tau) - \left(\xi_{\chi_\varepsilon} \right)_0(l/\varepsilon) \right|^p e^{2\pi\delta(-\tau+\tau(\varepsilon))} d\tau dt \\
& \leq C \varepsilon^{p\gamma} e^{2\pi(p-1)S(\varepsilon)} \int_{S^1} \int_0^{\frac{p-1}{\delta}S(\varepsilon)} s^{p\gamma} e^{-2\pi\delta s} ds dt \cdot \left\| \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{W_\varepsilon^{1,p}([l/\varepsilon, \tau(\varepsilon)])}^p \\
& \leq C(p, l) \left\| \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{W_\varepsilon^{1,p}([l/\varepsilon, \tau(\varepsilon)])}^p,
\end{aligned}$$

where $s = \tau - \frac{l}{\varepsilon}$ and in the last inequality we have used that the integral $\int_0^\infty s^{p\gamma} e^{-2\pi\delta s} ds$ converges and $\varepsilon^{p\gamma} e^{2\pi(p-1)S(\varepsilon)} \approx l^{p-1}$. From linearized gradient operator $\frac{\partial}{\partial\tau} + A_\varepsilon(\tau)$ we also have

$$\nabla_\tau \left(\left(\xi_{\chi_\varepsilon} \right)_0(\tau) - v_+ \right) = \nabla_\tau \left(\xi_{\chi_\varepsilon} \right)_0(\tau) = \left(\eta_{\chi_\varepsilon} \right)_0 - A_\varepsilon(\tau) \left(\xi_{\chi_\varepsilon} \right)_0.$$

Since $\left(\xi_{\chi_\varepsilon} \right)_0(\tau) - v_+$ is t -independent, and from (12.3) for $\tau \in [l/\varepsilon, \tau(\varepsilon)]$ we have

$$\begin{aligned}
& \left\| \nabla \left(\left(\xi_{\chi_\varepsilon} \right)_0(\tau) - v_+ \right) e^{-\frac{2\pi\delta}{p}(\tau-\tau(\varepsilon))} \right\|_{L^p([l/\varepsilon, \tau(\varepsilon)])} \\
& \leq C \left\| \nabla_\tau \left(\left(\xi_{\chi_\varepsilon} \right)_0(\tau) - v_+ \right) \varepsilon^{\frac{1-p}{p}} \right\|_{L^p([l/\varepsilon, \tau(\varepsilon)])} \\
& = C \left\| \nabla_\tau \left(\xi_{\chi_\varepsilon} \right)_0(\tau) \right\|_{L_\varepsilon^p([l/\varepsilon, \tau(\varepsilon)])} \\
& \leq C \left(\left\| \left(\eta_{\chi_\varepsilon} \right)_0 \right\|_{L_\varepsilon^p([l/\varepsilon, \tau(\varepsilon)])} + \|A_\varepsilon(\tau)\|_{C^0} \left\| \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{L_\varepsilon^p([l/\varepsilon, \tau(\varepsilon)])} \right) \\
& \leq C \left(\left\| \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{W_\varepsilon^{1,p}(\mathbb{R})} + \left\| \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{W_\varepsilon^{1,p}([l/\varepsilon, \tau(\varepsilon)])} \right).
\end{aligned}$$

Note in the last inequality we must use $\left\| \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{W_\varepsilon^{1,p}(\mathbb{R})}$ instead of

$$\left\| \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{W_\varepsilon^{1,p}([l/\varepsilon, \tau(\varepsilon)])}$$

since we have used the right inverse defined on whole \mathbb{R} . Combining these we get

$$\left\| \left(\left(\xi_{\chi_\varepsilon} \right)_0(\tau) - v_+ \right) \right\|_{W_\varepsilon^{1,p}([l/\varepsilon, \tau(\varepsilon)])} \leq C \left\| \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{W_\varepsilon^{1,p}(\mathbb{R})}. \tag{12.5}$$

In (12.4) and (12.5), the interval \mathbb{R} is bigger than $[-l/\varepsilon, l/\varepsilon]$ where ξ_{χ_ε} was originally defined. However, we will prove the following inequality

$$\begin{aligned} \left\| \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{W_\varepsilon^{1,p}(\mathbb{R})} &\leq C \left\| \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{W_{\rho_\varepsilon}^{1,p}([-l/\varepsilon, l/\varepsilon])}, \\ \left\| \widetilde{\xi_{\chi_\varepsilon}} \right\|_{W_{\rho_\varepsilon}^{1,p}(\mathbb{R} \times S^1)}^p &\leq C \left\| \widetilde{\xi_{\chi_\varepsilon}} \right\|_{W_{\rho_\varepsilon}^{1,p}([-l/\varepsilon, l/\varepsilon] \times S^1)}^p, \end{aligned} \quad (12.6)$$

later in Lemma 12.2 and 12.3 where C is independent on ε . Therefore combining (12.1), (12.2), (12.4) and (12.5), we have

$$\|\xi_\varepsilon - (\xi_\varepsilon)_0(l/\varepsilon)\|_{W_{\beta_{\delta,\varepsilon}}^{1,p}([l/\varepsilon, \tau(\varepsilon)] \times S^1)} \leq C \left(\|\xi_+\|_{W_\alpha^{1,p}(\Sigma_+)} + \|\xi_{\chi_\varepsilon}\|_{W_{\rho_\varepsilon}^{1,p}([-l/\varepsilon, l/\varepsilon] \times S^1)} \right).$$

The estimate for $\tau \in [-\tau(\varepsilon), 0]$ is similar.

For $|\tau| > \tau(\varepsilon)$, $\xi_\varepsilon = \xi_\pm^\varepsilon$ is a shift of ξ_\pm , hence

$$\left\| \widetilde{\xi_\varepsilon} \right\|_{W_{\beta_{\delta,\varepsilon}}^{1,p}(\pm[\tau(\varepsilon), \infty] \times S^1)} = \left\| \widetilde{\xi_\pm} \right\|_{W_{\beta_{\delta,\varepsilon}}^{1,p}(\pm[0, \infty] \times S^1)} \leq \|\xi_\pm\|_{W_\alpha^{1,p}(\Sigma_\pm)}.$$

Combining these we have

$$\begin{aligned} \|\xi_\varepsilon\|_\varepsilon &= \left\| \widetilde{\xi_\varepsilon} \right\|_{W_{\beta_{\delta,\varepsilon}}^{1,p}(\mathbb{R} \times S^1)} + \|(\xi_\varepsilon)_0\|_{W_\varepsilon^{1,p}([-l/\varepsilon, l/\varepsilon])} + |(\xi_\varepsilon)_0(\pm l/\varepsilon)| \\ &\leq C \left(\|\xi_+\|_{W_\alpha^{1,p}(\Sigma_+)} + \|\xi_-\|_{W_\alpha^{1,p}(\Sigma_-)} + \|\xi_{\chi_\varepsilon}\|_{W_{\rho_\varepsilon}^{1,p}([-l/\varepsilon, l/\varepsilon] \times S^1)} \right) + C\varepsilon|\mu| \\ &\leq C \left(\|\eta_+\|_{L_\alpha^p(\Sigma_+)} + \|\eta_-\|_{L_\alpha^p(\Sigma_-)} + \|\eta_{\chi_\varepsilon}\|_{L_{\rho_\varepsilon}^p([-l/\varepsilon, l/\varepsilon] \times S^1)} \right) \\ &= C \left(\|\kappa_+^\varepsilon \eta\|_{L_\alpha^p(\Sigma_+)} + \|\kappa_-^\varepsilon \eta\|_{L_\alpha^p(\Sigma_-)} + \|\kappa_0^\varepsilon \eta\|_{L_{\rho_\varepsilon}^p([-l/\varepsilon, l/\varepsilon] \times S^1)} \right) \\ &\leq C \|\eta\|_\varepsilon, \end{aligned}$$

Thus we have obtained

$$\|Q_{para}^{app;\varepsilon} \eta\|_\varepsilon = \|(\xi_\varepsilon, \mu)\|_\varepsilon \leq C \|\eta\|_\varepsilon.$$

The proposition is now proved. \square

Finally it remains to prove (12.6) which is in order.

Lemma 12.2. *We have*

$$\left\| \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{W_\varepsilon^{1,p}(\mathbb{R})} \leq C \left\| \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{W_{\rho_\varepsilon}^{1,p}([-l/\varepsilon, l/\varepsilon])}.$$

Proof. Since the right inverse $L_\varepsilon^p(\mathbb{R}) \rightarrow W_\varepsilon^{1,p}(\mathbb{R})$, $(\eta_{\chi_\varepsilon})_0 \rightarrow (\xi_{\chi_\varepsilon})_0$ is uniformly bounded, and $(\eta_{\chi_\varepsilon})_0$ is supported in $[-l/\varepsilon, l/\varepsilon]$, we have

$$\begin{aligned} \left\| \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{W_\varepsilon^{1,p}(\mathbb{R})} &\leq C \left\| \left(\eta_{\chi_\varepsilon} \right)_0 \right\|_{L_\varepsilon^p(\mathbb{R})} = C \left\| \left(\eta_{\chi_\varepsilon} \right)_0 \right\|_{L_\varepsilon^p([-l/\varepsilon, l/\varepsilon])} \\ &= C \left\| D^\varepsilon \circ \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{L_\varepsilon^p([-l/\varepsilon, l/\varepsilon])} \leq C \left\| \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{W_\varepsilon^{1,p}([-l/\varepsilon, l/\varepsilon])} \\ &\leq C \left\| \left(\xi_{\chi_\varepsilon} \right)_0 \right\|_{W_{\rho_\varepsilon}^{1,p}([-l/\varepsilon, l/\varepsilon])}, \end{aligned}$$

where the last inequality is because $W_\varepsilon^{1,p}$ is a component of $W_{\rho_\varepsilon}^{1,p}$. \square

Similarly we prove the following

Lemma 12.3. *We have*

$$\left\| \widetilde{\xi_{\chi_\varepsilon}} \right\|_{W_{\beta_{\delta,\varepsilon}}^{1,p}(\mathbb{R} \times S^1)} \leq C \left\| \nabla \widetilde{\xi_{\chi_\varepsilon}} \right\|_{L_{\beta_{\delta,\varepsilon}}^p([-l/\varepsilon, l/\varepsilon] \times S^1)}.$$

Proof. By previous proposition, Q_{app}^ε is a uniformly bounded operator, so for the higher mode $\widetilde{\xi_{\chi_\varepsilon}}$ of ξ_{χ_ε} , we have

$$\begin{aligned} \left\| \widetilde{\xi_{\chi_\varepsilon}} \right\|_{W_{\beta_{\delta,\varepsilon}}^{1,p}(\mathbb{R} \times S^1)} &\leq C \left\| D_{para}^\varepsilon \widetilde{\xi_{\chi_\varepsilon}} \right\|_{L_{\beta_{\delta,\varepsilon}}^p(\mathbb{R} \times S^1)} \\ &= C \left\| \kappa_0^\varepsilon \widetilde{\eta_{\chi_\varepsilon}} \right\|_{L_{\beta_{\delta,\varepsilon}}^p(\mathbb{R} \times S^1)} \\ &= C \left\| D^\varepsilon \widetilde{\xi_{\chi_\varepsilon}} \right\|_{L_{\beta_{\delta,\varepsilon}}^p([-l/\varepsilon, l/\varepsilon] \times S^1)} \\ &\leq C \left\| \nabla \widetilde{\xi_{\chi_\varepsilon}} \right\|_{L_{\beta_{\delta,\varepsilon}}^p([-l/\varepsilon, l/\varepsilon] \times S^1)}. \end{aligned}$$

\square

This finishes the proof of Proposition 12.1, which establishes the construction of the right inverse with uniform bound as $\varepsilon \rightarrow 0$ for $L \geq l \geq l_0$ for any given $L, l_0 > 0$.

We now justify that $Q_{para}^{app;\varepsilon}$ is indeed an approximate right inverse.

Proposition 12.4. $\left\| (D_{para}^\varepsilon \circ Q_{para}^{app;\varepsilon} - I) \eta \right\|_\varepsilon < \frac{1}{2} \|\eta\|_\varepsilon$.

Proof. By the definition of $Q_{para}^{app;\varepsilon}$, $(D_{para}^\varepsilon \circ Q_{para}^{app;\varepsilon} - I) \eta = 0$ except on the intervals $\pm [l/\varepsilon - T(\varepsilon), l/\varepsilon + T(\varepsilon)]$. Let's consider $\tau \in [l/\varepsilon - T(\varepsilon), l/\varepsilon + T(\varepsilon)]$. The other interval is the same. Recall $\widetilde{\xi_{\chi_\varepsilon}} = \xi_{\chi_\varepsilon} - \left(\xi_{\chi_\varepsilon} \right)_0$, $\widetilde{\xi_+} = \xi_+ - v_+$ and

$$\xi_\varepsilon = \left(\xi_{\chi_\varepsilon} \right)_0 + \phi_0^{l/\varepsilon + T(\varepsilon)} (\tau) \widetilde{\xi_{\chi_\varepsilon}} + \phi_+^{(l/\varepsilon - T(\varepsilon))} (\tau) \widetilde{\xi_+}.$$

We compute

$$\begin{aligned} &D_{para}^\varepsilon \circ Q_{para}^{app;\varepsilon} \eta - \eta \\ &= D_{para}^\varepsilon \left(\left(\xi_{\chi_\varepsilon} \right)_0, \mu \right) + \phi_0^{l/\varepsilon + T(\varepsilon)} D_{para}^\varepsilon \widetilde{\xi_{\chi_\varepsilon}} + \phi_+^{l/\varepsilon - T(\varepsilon)} D_{para}^\varepsilon \widetilde{\xi_+} - (\eta_0 + \widetilde{\eta}) \\ &\quad + \left(\phi_0^{l/\varepsilon + T(\varepsilon)} \right)' \widetilde{\xi_{\chi_\varepsilon}} + \left(\phi_+^{l/\varepsilon - T(\varepsilon)} \right)' \widetilde{\xi_+} \\ &= D_{para}^\varepsilon \left(\left(\xi_{\chi_\varepsilon} \right)_0, \mu \right) + \phi_0^{l/\varepsilon + T(\varepsilon)} D_{para}^\varepsilon \widetilde{\xi_{\chi_\varepsilon}} + \phi_+^{l/\varepsilon - T(\varepsilon)} \left[D_{u_+}^\varepsilon \overline{\partial}_{(K_+, J_+)} + A_\varepsilon \right] (\xi_+^\varepsilon - v_+) \\ &\quad - (\eta_0 + \widetilde{\eta}) + \left(\phi_0^{l/\varepsilon + T(\varepsilon)} \right)' \widetilde{\xi_{\chi_\varepsilon}} + \left(\phi_+^{l/\varepsilon - T(\varepsilon)} \right)' \widetilde{\xi_+} \\ &= D_{para}^\varepsilon \left(\left(\xi_{\chi_\varepsilon} \right)_0, \mu \right) + \phi_0^{l/\varepsilon + T(\varepsilon)} D_{para}^\varepsilon \widetilde{\xi_{\chi_\varepsilon}} + \phi_+^{l/\varepsilon - T(\varepsilon)} D_{u_+}^\varepsilon \overline{\partial}_{(K_+, J_+)} \xi_+^\varepsilon - (\eta_0 + \widetilde{\eta}) \\ &\quad + \phi_+^{l/\varepsilon - T(\varepsilon)} \left(A_\varepsilon \widetilde{\xi_+} - D_{u_+}^\varepsilon \overline{\partial}_{(K_+, J_+)} v_+ \right) + \left(\phi_0^{l/\varepsilon + T(\varepsilon)} \right)' \widetilde{\xi_{\chi_\varepsilon}} + \left(\phi_+^{l/\varepsilon - T(\varepsilon)} \right)' \widetilde{\xi_+}, \end{aligned}$$

where in the second identity we have used the notation $\widetilde{\xi_+} = \xi_+^\varepsilon - v_+$. By our construction

$$D_{para}^\varepsilon \left(\left(\xi_{\chi_\varepsilon} \right)_0, \mu \right) = \eta_0, \quad D_{para}^\varepsilon \widetilde{\xi_{\chi_\varepsilon}} = \kappa_0^\varepsilon \widetilde{\eta}, \quad \text{and} \quad D_{u_+}^\varepsilon \overline{\partial}_{(K_+, J_+)} \xi_+^\varepsilon = \kappa_+^\varepsilon \widetilde{\eta}$$

in $[l/\varepsilon - T(\varepsilon), l/\varepsilon + T(\varepsilon)]$. Then substitution of these into the above and canceling out makes the second to the last row in the above identity become 0.

For the term $A_\varepsilon(\tau) \widetilde{\xi}_+$, using $\|A_\varepsilon\|_{C^1} \leq C\varepsilon$, we obtain

$$\begin{aligned} \left\| \phi_+^{l/\varepsilon - T(\varepsilon)}(\tau) A_\varepsilon(\tau) \widetilde{\xi}_+ \right\|_\varepsilon &\leq C\varepsilon \left\| \widetilde{\xi}_+ \right\|_\varepsilon \leq C\varepsilon \left\| \xi_+ \right\|_\varepsilon \\ &\leq C\varepsilon \left\| \eta_+ \right\|_\varepsilon \leq C\varepsilon \left\| \eta \right\|_\varepsilon, \end{aligned}$$

where the first inequality holds because ξ_+^ε is a shift of ξ_+ , the second holds by the definition of the norm $\|\cdot\|_\varepsilon$ and the third comes from the boundedness of the right inverse of $D_{u_+} \bar{\partial}_{(K_+, J_+)}$.

For $D_{u_+} \bar{\partial}_{(K_+^\varepsilon, J_+^\varepsilon)} v_+$, by the fact that on

$$\tau \in [l/\varepsilon - T(\varepsilon) - \tau(\varepsilon), l/\varepsilon + T(\varepsilon) - \tau(\varepsilon)] \subset [-\infty, -1],$$

$J_+ = J_0$, and v_+ is a vector field obtained by parallel transport of $\xi_+(o_+)$,

$$|v_+| = |\xi_+(o_+)|, |\nabla v_+(o_+)| = 0$$

and

$$\begin{aligned} |D_{u_+} \bar{\partial}_{(K_+, J_+)} v_+| &= \left| (\nabla v_+)^{0,1} + \frac{1}{2} J_0 \nabla_{v_+} J_0 \partial u_+ \right| \\ &\leq C |du_+| |v_+| = C |du_+| |\xi_+(o_+)|. \end{aligned}$$

Using the uniform exponential decay $|du_+| \leq Ce^{2\pi\tau}$ as $\tau \rightarrow -\infty$, and noticing $\varepsilon^{1-p} \leq e^{-2\pi\delta(\tau - \tau(\varepsilon))}$ for $\tau \in [l/\varepsilon - T(\varepsilon), l/\varepsilon]$, we have

$$|du_+(\tau)| \leq Ce^{2\pi(-l/\varepsilon + T(\varepsilon))}$$

for $\tau \in [l/\varepsilon - T(\varepsilon), l/\varepsilon]$, hence

$$\begin{aligned} \|D_{u_+} \bar{\partial}_{(K_+, J_+)} v_+\|_\varepsilon &\leq C |du_+(\tau)| |\xi_+(o_+)| \\ &\leq Ce^{2\pi(-l/\varepsilon + T(\varepsilon))} \|\xi_+\|_{W_\alpha^{1,p}(\Sigma_+)} \\ &\leq Ce^{2\pi(-l/\varepsilon + T(\varepsilon))} \|\eta\|_{L_\alpha^p(\Sigma_+)}. \end{aligned}$$

Here we have used that $|\xi_+(o_+)|$ is part of the norm $\|\xi_+\|_{W_\alpha^{1,p}(\Sigma_+)}$.

We estimate the remaining two terms

$$\left(\phi_0^{l/\varepsilon + T(\varepsilon)} \right)' \widetilde{\xi}_{\chi_\varepsilon}, \quad \left(\phi_+^{l/\varepsilon - T(\varepsilon)} \right)' \widetilde{\xi}_+.$$

For $\left(\phi_+^{l/\varepsilon - T(\varepsilon)} \right)'(\tau) \widetilde{\xi}_+$, it is supported in $[l/\varepsilon - T(\varepsilon), l/\varepsilon - T(\varepsilon) + 1]$, where both the power order weight and ε -adiabatic weight are dominated by the exponential weight as the following

$$\varepsilon^{1-p+\delta} (1 + |\tau|)^\delta \leq \varepsilon^{1-p} \leq e^{-2\pi\delta T(\varepsilon)} \cdot e^{2\pi\delta|\tau - \tau(\varepsilon)|}.$$

Therefore we do not need to separate the 0-mode and the higher mode parts of $\phi'_{l/\varepsilon-T(\varepsilon)}(\xi_+ - v_+)$ and immediately see

$$\begin{aligned} & \left\| \left(\phi_+^{l/\varepsilon-T(\varepsilon)} \right)'(\tau) \widetilde{\xi}_+^\varepsilon \right\|_\varepsilon \\ & \leq e^{\frac{-2\pi\delta T(\varepsilon)}{p}} \left\| \left(\phi_+^{l/\varepsilon-T(\varepsilon)} \right)'(\tau) \widetilde{\xi}_+^\varepsilon \right\|_{W_\delta^{1,p}([l/\varepsilon-T(\varepsilon), l/\varepsilon-T(\varepsilon)+1])} \\ & \leq e^{\frac{-2\pi\delta T(\varepsilon)}{p}} \|\xi_+\|_{W_\alpha^{1,p}(\Sigma_+)} \leq e^{\frac{-2\pi\delta T(\varepsilon)}{p}} C \|\eta_+\|_{L_\alpha^p(\Sigma_+)} \leq C e^{\frac{-2\pi\delta T(\varepsilon)}{p}} \|\eta\|_\varepsilon. \end{aligned}$$

For $\left(\phi_+^{l/\varepsilon-T(\varepsilon)} \right)' \widetilde{\xi}_+^\varepsilon$, it is supported in $[l/\varepsilon + T(\varepsilon), l/\varepsilon + T(\varepsilon) + 1]$. From the Sobolev embedding, we obtain

$$\begin{aligned} & \left| \widetilde{\xi}_+^\varepsilon \right| \\ & \leq C \varepsilon^{\frac{p-1-\delta}{p}} (1 + l/\varepsilon + T(\varepsilon))^{-\frac{\delta}{p}} \left\| \widetilde{\xi}_+^\varepsilon \right\|_{W_{\rho_\varepsilon}^{1,p}([l/\varepsilon+T(\varepsilon), l/\varepsilon+T(\varepsilon)+1] \times S^1)} \\ & \leq C \varepsilon^{\frac{p-1-\delta}{p}} (\varepsilon/l)^{\frac{\delta}{p}} \left\| \widetilde{\xi}_+^\varepsilon \right\|_{W_{\rho_\varepsilon}^{1,p}([l/\varepsilon+T(\varepsilon), l/\varepsilon+T(\varepsilon)+1] \times S^1)} \\ & \leq C \varepsilon^{\frac{p-1-\delta}{p}} (\varepsilon/l)^{\frac{\delta}{p}} \cdot C \left\| \tilde{\eta}_{\chi_\varepsilon} \right\|_{L_{\rho_\varepsilon}^p(\mathbb{R} \times S^1)} \\ & \leq C \varepsilon^{\frac{p-1-\delta}{p}} (\varepsilon/l)^{\frac{\delta}{p}} \|\eta\|_\varepsilon, \end{aligned}$$

Since $[l/\varepsilon + T(\varepsilon), l/\varepsilon + T(\varepsilon) + 1]$ is contained outside $[-l/\varepsilon, l/\varepsilon]$, we needn't to distinguish its 0-mode and higher mode in computing the weighted Sobolev norm of $\left(\phi_+^{l/\varepsilon-T(\varepsilon)} \right)'(\tau) \widetilde{\xi}_+^\varepsilon$. Note

$$\left(\phi_+^{l/\varepsilon-T(\varepsilon)} \right)'(\tau) \widetilde{\xi}_+^\varepsilon \Big|_{\tau=l/\varepsilon} = \left(\phi_+^{l/\varepsilon-T(\varepsilon)} \right)'(\tau) \left(\xi_{\chi_\varepsilon} - (\xi_{\chi_\varepsilon})_0 \right) \Big|_{\tau=l/\varepsilon} = 0,$$

and the weight there is

$$e^{-2\pi\delta(\tau-l/\varepsilon-\frac{p-1}{\delta}S(\varepsilon))} \leq e^{-2\pi\delta(T(\varepsilon)-\frac{p-1}{\delta}S(\varepsilon))} = (1 + l/\varepsilon)^{p-1} e^{-2\pi\delta T(\varepsilon)},$$

therefore

$$\begin{aligned} \left\| \left(\phi_+^{l/\varepsilon-T(\varepsilon)} \right)'(\tau) \widetilde{\xi}_+^\varepsilon \right\|_\varepsilon & \leq C \varepsilon^{\frac{p-1-\delta}{p}} (\varepsilon/l)^{\frac{\delta}{p}} \|\eta\|_\varepsilon \cdot (1 + l/\varepsilon)^{\frac{p-1}{p}} e^{\frac{-2\pi\delta T(\varepsilon)}{p}} \\ & = C e^{\frac{-2\pi\delta T(\varepsilon)}{p}} \|\eta\|_\varepsilon. \end{aligned}$$

Combining these estimates we obtain

$$\begin{aligned} \|D_{para}^\varepsilon \bar{\partial}_{(K_\varepsilon, J_\varepsilon)} \circ Q_{para}^{app;\varepsilon}(\eta) - \eta\|_\varepsilon & \leq C \varepsilon \|\eta\|_\varepsilon + C e^{\frac{-2\pi\delta T(\varepsilon)}{p}} \|\eta\|_\varepsilon + C e^{\frac{-2\pi\delta T(\varepsilon)}{p}} \|\eta\|_\varepsilon \\ & \quad + C e^{2\pi(-l/\varepsilon+T(\varepsilon))} \|\eta\|_{L_\alpha^p(\Sigma_+)} \end{aligned}$$

Since $T(\varepsilon) = \frac{p-1}{3\delta}S(\varepsilon) \rightarrow \infty$ and $\varepsilon S(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, we have for small enough ε ,

$$\|D_{para}^\varepsilon \bar{\partial}_{(K_\varepsilon, J_\varepsilon)} \circ Q_{para}^{app;\varepsilon}(\eta) - \eta\|_\varepsilon \leq \frac{1}{2} \|\eta\|_\varepsilon.$$

The proposition follows. \square

By the above proposition, $D_{para}^\varepsilon \bar{\partial}_{(K_\varepsilon, J_\varepsilon)} \circ Q_{para}^{app;\varepsilon}$ is invertible, and

$$\begin{aligned} \left\| (D_{para}^\varepsilon \bar{\partial}_{(K_\varepsilon, J_\varepsilon)} \circ Q_{para}^{app;\varepsilon})^{-1} \right\| &\leq \left\| \sum_{k=0}^{\infty} (D_{para}^\varepsilon \bar{\partial}_{(K_\varepsilon, J_\varepsilon)} \circ Q_{para}^{app;\varepsilon} - id)^k \right\| \\ &\leq \sum_{k=0}^{\infty} \left(\frac{1}{2} \right)^k = 1, \end{aligned}$$

so we can construct the true right inverse of $D_{para}^\varepsilon \bar{\partial}_{(K_\varepsilon, J_\varepsilon)}$ as

$$Q_{para}^\varepsilon := Q_{para}^{app;\varepsilon} \circ (D_{para}^\varepsilon \bar{\partial}_{(K_\varepsilon, J_\varepsilon)} \circ Q_{para}^{app;\varepsilon})^{-1}.$$

Since $Q_{para}^{app;\varepsilon}$ is uniformly bounded, so is Q_{para}^ε .

Remark 12.5. If we examine the proof of the right inverse bound of Q_{para}^ε , we get $\|Q_{para}^\varepsilon\| \leq Cl^{\frac{p-1}{p}}$. It will increase as $l \rightarrow \infty$. But the $\bar{\partial}_{(K_\varepsilon, J_\varepsilon)}$ -error estimate will have faster order decay $\left(Ce^{-cl} + l^{-\frac{p-1}{\delta}} \right) l^{\frac{p-1}{p}} \varepsilon^{\frac{1}{p}}$, and the quadratic estimate in next section has the uniform constant C , so we can still apply the implicit function theorem for all $l \geq l_0$.

13. UNIFORM QUADRATIC ESTIMATE AND IMPLICIT FUNCTION THEOREM

Consider the Banach spaces

$$\begin{aligned} X &= \left\{ (\xi, \mu) \mid \xi \in \Gamma \left((u_{app}^\varepsilon)^* TM \right), \mu \in T_{l/\varepsilon} \mathbb{R}, \|(\xi, \mu)\|_\varepsilon = \|\xi\|_\varepsilon + |\mu| < \infty \right\} \\ Y &= \left\{ \eta \mid \eta \in \Gamma \left((u_{app}^\varepsilon)^* TM \right) \otimes \Lambda^{0,1}(\mathbb{R} \times S^1), \|\eta\|_\varepsilon < \infty \right\} \end{aligned}$$

with the Banach norms $\|\cdot\|_\varepsilon$ defined for ξ and η in section 9. For sections ξ, ξ' in $(u_{app}^\varepsilon)^* TM$ and μ, μ' in $T_{l/\varepsilon} \mathbb{R}$, let

$$\mathcal{F}_{u_{app}^\varepsilon} : \Gamma \left((u_{app}^\varepsilon)^* TM \right) \times T_{l/\varepsilon} \mathbb{R} \rightarrow \Gamma \left((u_{app}^\varepsilon)^* TM \right) \otimes \Lambda^{0,1}(\mathbb{R} \times S^1)$$

that

$$\mathcal{F}_{u_{app}^\varepsilon}(\xi, \mu)(\tau, t) = Pal_\xi^{-1} \left[\bar{\partial}_{(K_\varepsilon, J_\varepsilon)} \exp_{u_{app}^\varepsilon}(\xi) \right] (P_\mu \tau, t)$$

where Pal_ξ^{-1} is the parallel transport from $\exp_{u_{app}^\varepsilon} \xi(\tau, t)$ to $u_{app}^\varepsilon(\tau, t)$ along the shortest geodesic, and the reparametrization map

$$P_\mu : (\mathbb{R}; -l/\varepsilon, l/\varepsilon) \rightarrow (\mathbb{R}; -(l/\varepsilon - \mu), l/\varepsilon - \mu).$$

between marked real lines is

$$P_\mu(\tau) = \begin{cases} \tau + \mu & \text{for } \tau < -l/\varepsilon, \\ \frac{l}{l-\varepsilon\mu} \tau & \text{for } |\tau| \leq l/\varepsilon, \\ \tau - \mu & \text{for } \tau > l/\varepsilon, \end{cases}$$

The above definition of $\mathcal{F}_{u_{app}^\varepsilon}$ is slightly imprecise since P_μ is only piecewise differentiable and we need to smooth it to a sufficiently close diffeomorphism, but that

can be done. Then

$$\begin{aligned}
& d\mathcal{F}_{u_{app}^\varepsilon}(\xi, \mu)(\xi', \mu')(\tau, t) \\
&= \left. \frac{d}{ds} \right|_{s=0} \mathcal{F}_{u_{app}^\varepsilon}(\xi + s\xi', \mu + s\mu')(\tau, t) \\
&= \left. \frac{d}{ds} \right|_{s=0} Pal_{\xi+s\xi'}^{-1} \left[\bar{\partial}_{(K_\varepsilon, J_\varepsilon)} \exp_{u_{app}^\varepsilon}(\xi + s\xi') \right] (P_{\mu+s\mu'}(\tau), t) \\
&= \begin{cases} \left. \frac{d}{ds} \right|_{s=0} Pal_{\xi+s\xi'}^{-1} \left[\bar{\partial}_{(K_\varepsilon, J_\varepsilon)} \exp_{u_{app}^\varepsilon}(\xi + s\xi') \right] \left(\frac{l\tau}{l-\varepsilon(\mu+s\mu')}, t \right) & \text{if } |\tau| \leq l/\varepsilon \\ \left. \frac{d}{ds} \right|_{s=0} Pal_{\xi+s\xi'}^{-1} \left[\bar{\partial}_{(K_\varepsilon, J_\varepsilon)} \exp_{u_{app}^\varepsilon}(\xi + s\xi') \right] (\tau \pm (\mu + s\mu'), t) & \text{if } |\tau| > l/\varepsilon \end{cases}
\end{aligned}$$

and

$$d\mathcal{F}_{u_{app}^\varepsilon}(0, 0)(\xi', \mu') = D_{u_{app}^\varepsilon} \bar{\partial}_{(K_\varepsilon, J_\varepsilon)}(\xi', \mu').$$

Proposition 13.1. *For each given pair $(K_\varepsilon, J_\varepsilon)$, there exists uniform constants C (depending only on l_0 and p but independent of ε) and h_0 such that for all (ξ, μ) and (ξ', μ') in $\Gamma((u_{app}^\varepsilon)^* TM) \times T_{l/\varepsilon} \mathbb{R}$ with $0 \leq \|(\xi, \mu)\|_\varepsilon \leq h_0$,*

$$\left\| d\mathcal{F}_{u_{app}^\varepsilon}(\xi, \mu)(\xi', \mu') - D_{u_{app}^\varepsilon} \bar{\partial}_{(K_\varepsilon, J_\varepsilon)}(\xi', \mu') \right\|_\varepsilon \leq C \|(\xi, \mu)\|_\varepsilon \|(\xi', \mu')\|_\varepsilon.$$

Proof. We first consider the case when $\mu = \mu' = 0$. For $|\tau| \leq l/\varepsilon$, and any $t \in S^1$, we have

$$\begin{aligned}
|\xi(\tau, t)| &\leq |\xi(\tau, t) - \xi_0(\tau)| + |\xi_0(\tau)| \\
&\leq C \left\| \tilde{\xi} \right\|_{W_{\beta_{\delta, \varepsilon}}^{1, p}([-l/\varepsilon, l/\varepsilon] \times S^1)} + C \|\xi_0\|_{W_\varepsilon^{1, p}([-l/\varepsilon, l/\varepsilon] \times S^1)} \\
&\leq C \|\xi\|_\varepsilon,
\end{aligned}$$

where the second row is by $W^{1, p} \hookrightarrow C^0$ Sobolev embedding, and the facts that the weight $\beta_{\delta, \varepsilon}$ is nowhere less than 1 and the length $l \geq l_0 > 0$. For $l/\varepsilon < |\tau| \leq l/\varepsilon + \frac{p-1}{\delta} S(\varepsilon)$, again by Sobolev embedding we have

$$\begin{aligned}
|\xi(\tau, t)| &\leq |\xi(\tau, t) - \xi_0(o_\pm)| + |\xi_0(o_\pm)| \\
&\leq C \left\| \tilde{\xi} \right\|_{W_{\beta_{\delta, \varepsilon}}^{1, p}(\pm[l/\varepsilon, l/\varepsilon + \frac{p-1}{\delta} S(\varepsilon)] \times S^1)} + |\xi_0(o_\pm)| \\
&\leq C \|\xi\|_\varepsilon.
\end{aligned}$$

For $|\tau| > \tau(\varepsilon)$ the weight $\beta_{\delta, \varepsilon}(\tau)$ is 1 so

$$|\xi(\tau, t)| \leq C \|\xi\|_\varepsilon \tag{13.1}$$

is standard Sobolev embedding. In the above the constants C only depends on l_0 and p . Combining these we have the uniform Sobolev constant

$$C(l_0, p) := \sup_{\xi \neq 0} \frac{\|\xi\|_\infty}{\|\xi\|_\varepsilon},$$

for our Banach norm $\|\cdot\|_\varepsilon$ of all ε for $\xi \in ((u_{app}^\varepsilon)^* TM)$ on $\mathbb{R} \times S^1$. The point estimate in the proof of Proposition 3.5.3 in [MS] yields

$$\left| d\mathcal{F}_{u_{app}^\varepsilon}(\xi) \xi' - D_{u_{app}^\varepsilon} \bar{\partial}_{(J_0, \varepsilon f)} \xi' \right| \leq A (|du_{app}^\varepsilon| |\xi| |\xi'| + |\xi| |\nabla \xi'| + |\nabla \xi| |\xi'|).$$

Taking the p -th power and integrating $|\nabla \xi|$ and $|\nabla \xi'|$ over $\mathbb{R} \times S^1$ with respect to the weight $\beta_{\delta, \varepsilon}$, while using the Sobolev inequality (from the above definition of $C(l_0, p)$)

$$|\xi|_\infty \leq C(l_0, p) \|\xi\|_\varepsilon$$

for the terms $|\xi|$ and $|\xi'|$, we obtain the uniform quadratic estimate

$$\left\| d\mathcal{F}_{u_{app}^\varepsilon}(\xi) \xi' - D_{u_{app}^\varepsilon} \bar{\partial}_{(J, \varepsilon f)} \xi' \right\|_\varepsilon \leq C \|\xi\|_\varepsilon \|\xi'\|_\varepsilon$$

where C is dependent on l_0, p but independent on ε .

Then we include the μ and μ' in the quadratic estimate. For simplicity of notation we let

$$u_\xi = \exp_{u_{app}^\varepsilon} \xi$$

and write the pair $(u_{app}^\varepsilon, l/\varepsilon)$ to emphasize the parameter l in the construction of u_{app}^ε . Then for $|\tau| \leq l/\varepsilon$,

$$\begin{aligned} & \left\| d\mathcal{F}_{(u_{app}^\varepsilon, \frac{l}{\varepsilon})}(\xi, \mu) (\xi', \mu') - d\mathcal{F}_{(u_{app}^\varepsilon, \frac{l}{\varepsilon})}(0, 0) (\xi', \mu') \right\| \\ &= \left\| \left(d\mathcal{F}_{(u_{app}^\varepsilon, \frac{l}{\varepsilon} - \mu)}(\xi) \xi' + \frac{\mu' l \varepsilon \tau}{(l - \varepsilon \mu)^2} Pal_\xi^{-1} \frac{\partial u_\xi}{\partial \tau} \right) - \left(d\mathcal{F}_{(u_{app}^\varepsilon, \frac{l}{\varepsilon})}(0) \xi' + \frac{\mu' \varepsilon \tau}{l} \frac{\partial u}{\partial \tau} \right) \right\| \\ &\leq \left\| d\mathcal{F}_{(u_{app}^\varepsilon, \frac{l}{\varepsilon} - \mu)}(\xi) \xi' - d\mathcal{F}_{(u_{app}^\varepsilon, \frac{l}{\varepsilon} - \mu)}(0) \xi' \right\| \\ &\quad + \left\| d\mathcal{F}_{(u_{app}^\varepsilon, \frac{l}{\varepsilon} - \mu)}(0) \xi' - d\mathcal{F}_{(u_{app}^\varepsilon, \frac{l}{\varepsilon})}(0) \xi' \right\| + \left\| \frac{\mu' l \varepsilon \tau}{(l - \varepsilon \mu)^2} Pal_\xi^{-1} \frac{\partial u_\xi}{\partial \tau} - \frac{\mu' \varepsilon \tau}{l} \frac{\partial u}{\partial \tau} \right\| \\ &\leq C \|\xi\|_\varepsilon \|\xi'\|_\varepsilon + C |\mu| \|\xi'\|_\varepsilon + C |\mu'| \|\xi\|_\varepsilon, \end{aligned} \tag{13.2}$$

where in the last inequality, the first term is by the $\mu = \mu' = 0$ case, the second term holds because the change of the conformal structure on $[-l/\varepsilon, l/\varepsilon] \times S^1$ by μ affects $\bar{\partial}_{(K_\varepsilon, J_\varepsilon)}$ in a linear way, and the third term is by the property of exponential map, and

$$\left| \frac{l \varepsilon \tau}{(l - \varepsilon \mu)^2} - \frac{\varepsilon \tau}{l} \right| = \left| \frac{\varepsilon \tau}{l} \right| \left| \left(\frac{1}{1 - \frac{\varepsilon}{l} \mu} \right)^2 - 1 \right| \leq 1 \cdot \left| \left(\frac{1}{1 \pm \frac{\varepsilon}{l_0} h_0} \right)^2 - 1 \right| \leq C \frac{h_0 \varepsilon}{l_0}$$

for $|\varepsilon \tau| \leq l$, $|\mu| \leq h_0$ and $l \geq l_0 > 0$.

For $|\tau| > l/\varepsilon$, the estimate to get (13.2) is similar (actually simpler) since

$$d\mathcal{F}_{(u_{app}^\varepsilon, \frac{l}{\varepsilon})}(\xi, \mu) (\xi', \mu') = d\mathcal{F}_{(u_{app}^\varepsilon, \frac{l}{\varepsilon} - \mu)}(\xi) \xi' + \mu' Pal_\xi^{-1} \frac{\partial u_\xi}{\partial \tau}. \tag{13.3}$$

Clearly

$$\begin{aligned} C \|\xi\|_\varepsilon \|\xi'\|_\varepsilon + C |\mu| \|\xi'\|_\varepsilon + C |\mu'| \|\xi\|_\varepsilon &\leq C (\|\xi\|_\varepsilon + |\mu|) (\|\xi'\|_\varepsilon + |\mu'|) \\ &= C \|(\xi, \mu)\|_\varepsilon \|(\xi', \mu')\|_\varepsilon. \end{aligned}$$

so the proposition follows. \square

Remark 13.2. When $l \rightarrow 0$ the Sobolev constant $C(l_0, p)$ blows up so we can not get uniform constant C in the above quadratic estimate. Different argument (blowing up the target) is needed for gluing. This is the nodal Floer case and was treated in [OZ1]. When $l_0 \leq l \rightarrow \infty$ the constant C remains uniform.

To perturb u_{app}^ε to be a true solution of the Floer equation $\bar{\partial}_{(K_\varepsilon, J_\varepsilon)} u = 0$, we need the following abstract implicit function theorem in [MS]

Proposition 13.3. *Let X, Y be Banach spaces and U be an open set in X . The map $F : X \rightarrow Y$ is continuous differentiable. For $x_0 \in U$, $D := dF(x_0) : X \rightarrow Y$ is surjective and has a bounded linear right inverse $Q : Y \rightarrow X$, with $\|Q\| \leq C$. Suppose that there exists $h > 0$ such that $x \in B_h(x_0) \subset U$*

$$x \in B_h(x_0) \subset U \implies \|dF(x) - D\| \leq \frac{1}{2C}.$$

Suppose

$$\|F(x_0)\| \leq \frac{h}{4C},$$

then there exists a unique $x \in B_h(x_0)$ such that

$$F(x) = 0, \quad x - x_0 \in \text{Image } Q, \quad \|x - x_0\| \leq 2C \|F(x_0)\|.$$

For the remaining section, we will wrap up the gluing construction by identifying the corresponding Banach spaces X, Y and the nonlinear map F , the point x_0 and the right inverse Q for the purpose of applying this proposition.

For a fixed sufficiently small $\varepsilon_0 > 0$, let $\varepsilon \in (0, \varepsilon_0]$. For any u_{app}^ε , we choose Banach spaces

$$X = \left(\Gamma((u_{app}^\varepsilon)^* TM) \times T_t \mathbb{R}, \|\cdot\|_\varepsilon \right), \quad Y = \left(\Gamma(u_{app}^\varepsilon)^* TM \otimes \Lambda^{0,1}(\mathbb{R} \times S^1), \|\cdot\|_\varepsilon \right)$$

with the Banach norms $\|\cdot\|_\varepsilon$ for ξ and η defined in section 9. We choose

$$U_\varepsilon = \{\xi \in X \mid \|\xi\|_\varepsilon\} < C(\varepsilon_0), \quad x_0 = 0 \in U_\varepsilon$$

for a constant $C(\varepsilon_0)$ depending only on ε_0 such that $C(\varepsilon_0) \rightarrow 0$ as $\varepsilon_0 \rightarrow 0$. We emphasize that this constant $C(\varepsilon_0)$ does not depend on the choice of $0 < \varepsilon \leq \varepsilon_0$.

Let F be the map $\mathcal{F}_{u_{app}^\varepsilon}$ defined in the beginning of the section. By Proposition 12.1 and Proposition 13.1 the C, h in our case are uniform while $\|F(x_0)\| \leq C\varepsilon^{\frac{1}{p}}$, so we can apply Proposition 13.3 to find a perturbation (ξ, μ) such that

$$\bar{\partial}_{(K_\varepsilon, J_\varepsilon)}(\exp_{u_{app}^\varepsilon} \xi)(P_\mu(\tau), t) = 0 \quad (13.4)$$

i.e., $(\exp_{u_{app}^\varepsilon} \xi)(P_\mu(\tau), t)$ is a genuine solution for (1.1).

We denote

$$\begin{aligned} & \mathcal{M}^{dfd}(K_-, J_-; f, K_+, J_+; z_-, z_+; A_- \# A_+) \\ &= \mathcal{M}(K_-, J_-; z_-; A_-)_{ev_+} \times_{ev_0} \mathcal{M}(f; [0, \ell])_{ev_\ell} \times_{ev_-} \mathcal{M}(K_+, J_+; z_+; A_+) \end{aligned}$$

and

$$\mathcal{M}_{\leq \varepsilon_0}^{para}(K, J; z_-, z_+; A_- \# A_+) = \bigcup_{\varepsilon \in (0, \varepsilon_0]} \mathcal{M}^\varepsilon(K_\varepsilon, J_\varepsilon; z_-, z_+; A_- \# A_+).$$

We denote by the smooth map

$$\begin{aligned} & \text{Glue} : (0, \varepsilon_0] \times \mathcal{M}_{(0,1,1)}^{dfd}(K_-, J_-; f, K_+, J_+; z_-, z_+; A_- \# A_+) \\ & \rightarrow \bigcup_{0 < \varepsilon \leq \varepsilon_0} \mathcal{M}^\varepsilon(K_\varepsilon, J_\varepsilon; z_-, z_+; A_- \# A_+) \end{aligned} \quad (13.5)$$

the composition of preG followed by the map

$$u_{app}^\varepsilon \rightarrow \left(\exp_{u_{app}^\varepsilon} \xi \right) (P_\mu \cdot, \cdot)$$

obtained by solving the equation (13.4) applying Proposition 13.3. We also denote by Glue_ε the slice of Glue for ε .

The main gluing theorem is the following

Theorem 13.4. *Let $(K_\varepsilon, J_\varepsilon)$ be the family of Floer data defined in (1.2). Then*

- (1) *there exists a topology on*

$$\mathcal{M}_{\leq \varepsilon_0}^{\text{para}}(K, J; z_-, z_+; A_- \# A_+) = \bigcup_{0 < \varepsilon \leq \varepsilon_0} \mathcal{M}^\varepsilon(K_\varepsilon, J_\varepsilon; z_-, z_+; A_- \# A_+)$$

with respect to which the gluing construction defines a proper embedding

$$\begin{aligned} \text{Glue} &: (0, \varepsilon_0] \times \mathcal{M}_{(0;1,1)}^{\text{dfd}}(K_-, J_-; f, K_+, J_+; z_-, z_+; A_- \# A_+) \\ &\rightarrow \mathcal{M}_{\leq \varepsilon_0}^{\text{para}}(K, J; z_-, z_+; A_- \# A_+) \end{aligned}$$

for sufficiently small ε_0 .

- (2) *the above mentioned topology can be embedded into*

$$\mathcal{M}^{\text{dfd}}(K_-, J_-; f, K_+, J_+; z_-, z_+; A_- \# A_+) \bigcup \mathcal{M}_{\leq \varepsilon_0}^{\text{para}}(K, J; z_-, z_+; A_- \# A_+)$$

as a set,

- (3) *the embedding Glue smoothly extends to the embedding*

$$\begin{aligned} \overline{\text{Glue}} &: [0, \varepsilon_0] \times \mathcal{M}^{\text{dfd}}(K_-, J_-; f, K_+, J_+; z_-, z_+; A_- \# A_+) \\ &\rightarrow \overline{\mathcal{M}_{\leq \varepsilon_0}^{\text{para}}}(K, J; z_-, z_+; A_- \# A_+) \end{aligned}$$

that satisfies $\overline{\text{Glue}}(0, u_-, \chi, u_+) = (u_-, \chi, u_+)$.

One essential ingredient to establish to complete the proof of this theorem is the following surjectivity whose proof we give in the next section.

Proposition 13.5. *There exists some $\varepsilon_0 > 0$, $\zeta_0 > 0$ and a function $\delta : (0, \varepsilon_0) \rightarrow \mathbb{R}_+$ such that the gluing map*

$$\text{Glue} : (0, \varepsilon_0] \times \mathcal{M}^{\text{dfd}}(K_-, J_-; f, K_+, J_+; z_-, z_+; A_- \# A_+)$$

is surjective onto

$$\mathcal{M}_{\leq \varepsilon_0}^{\text{para}}(K, J; z_-, z_+; A_- \# A_+) \cap \bigcup_{\substack{0 < \varepsilon \leq \varepsilon_0 \\ 0 < \zeta \leq \zeta_0}} V_{\zeta, \delta(\varepsilon)}^\varepsilon.$$

Here $V_{\zeta, \delta}^\varepsilon$ is the open subset given in (4.3).

Once this is established, the standard argument employed by Donaldson will complete the proof of this main theorem.

14. SURJECTIVITY

In this section, we give the proof of Proposition 13.5.

To prove surjectivity, we need to show that there exists $\varepsilon_0 > 0$, $0 < \zeta_0 < 1$ and a function $\delta : (0, \varepsilon_0] \rightarrow \mathbb{R}_+$ such that for $0 < \varepsilon \leq \varepsilon_0$ and $0 < \zeta \leq \zeta_0$, any pair $(\varepsilon, u) \in \mathcal{M}_{\leq \varepsilon_0}^{\text{para}}(K, J; z_-, z_+; A_- \# A_+)$ with

$$d_{\text{adia}}^{\varepsilon, \zeta}(u, (u_-, \chi, u_+)) < \delta(\varepsilon) \tag{14.1}$$

lies in the image of the gluing map

$$\text{Glue}_\varepsilon : \mathcal{M}^{\text{dfd}}(K_-, J_-; f, K_+, J_+; z_-, z_+; A_- \# A_+) \rightarrow \mathcal{M}(K_\varepsilon, J_\varepsilon; z_-, z_+; A_- \# A_+).$$

The above condition (14.1) implies that

- (1) $E_{J, \Theta_\varepsilon}(u) < \delta(\varepsilon)$,
- (2) $d_H(u([-R(\varepsilon), R(\varepsilon)] \times S^1), \chi([-l, l])) < \delta(\varepsilon)$,
- (3) $d_{C^\infty}(\pm[\frac{1}{2\pi} \ln \zeta_0, \infty) \times S^1)(u(\cdot \pm \tau(\varepsilon), \cdot), u_\pm) < \delta(\varepsilon)$,
- (4) $\text{diam}(u(\pm[R(\varepsilon), \tau(\varepsilon) + \frac{1}{2\pi} \ln \zeta_0] \times S^1)) < \delta(\varepsilon)$.

By definition of $V_{\zeta, \delta}^\varepsilon$, any element $u \in V_{\zeta, \delta}^\varepsilon$ can be expressed as

$$u(\tau, t) = \exp_{u_{app}^\varepsilon}(\xi)(\tau, t) \quad (14.2)$$

with $u_{app}^\varepsilon = \text{preG}(\varepsilon, u_-, \chi, u_+)$ for some

$$(u_-, \chi, u_+) \in \mathcal{M}^{dfd}(K_-, J_-; f, K_+, J_+; z_-, z_+; A_- \# A_+)$$

and $\xi \in \Gamma((u_{app}^\varepsilon)^* TM)$ for some ξ . More precisely we introduce the off-shell version of the pre-gluing map

$$\widetilde{\text{preG}} : ((u_-, \xi_-), (\chi, \xi_0), (u_+, \xi_+), \mu) \mapsto \text{preG}(\exp_{u_-}(\xi_-), \exp_\chi(\xi_0), \exp_{u_+}(\xi_+))(P_\mu \tau, t)$$

by the same formula for preG as (8.4) with u_\pm and χ replaced by $\exp_{u_\pm}(\xi_\pm)$ and $\exp_\chi(\xi_0)$ respectively, and $\mu \in T_{R(\varepsilon)}\mathbb{R}_+$ corresponds to the reparameterization of u_{app}^ε for $(\tau, t) \in [-R(\varepsilon), R(\varepsilon)] \times S^1$ by $(\tau', t) \in [-R(\varepsilon) + \mu, R(\varepsilon) - \mu] \times S^1$ as in (9.5).

The following is an immediate translation of the stable map convergence together with adiabatic convergence result in Theorem 4.3.

Lemma 14.1. *There exists some $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ any*

$$u \in \mathcal{M}(K_\varepsilon, J_\varepsilon; z_-, z_+; A_- \# A_+) \cap \bigcup_{0 < \zeta \leq \zeta_0} V_{\zeta, \delta(\varepsilon)}^\varepsilon,$$

can be expressed as

$$u(\tau, t) = \widetilde{\text{preG}}((u_-, \xi_-), (\chi, \xi_0), (u_+, \xi_+))(P_\mu \tau, t)$$

for some (u_-, χ, u_+) such that

$$\|\xi_\pm\|_{L^\infty} \leq \delta, \quad \|\xi_0\|_{L^\infty} \leq \delta, \quad |\mu| \leq \delta. \quad (14.3)$$

By the above lemma, for $u \in \mathcal{M}(K_\varepsilon, J_\varepsilon; z_-, z_+; A_- \# A_+) \cap \bigcup_{0 < \zeta \leq \zeta_0} V_{\zeta, \delta(\varepsilon)}^\varepsilon$ with $0 < \varepsilon \leq \varepsilon_0$, we can represent u by tangent vectors $(\xi_-, \xi_0, \xi_+, \mu)$ via

$$u(\tau, t) = \widetilde{\text{preG}}((u_-, \xi_-), (\chi, \xi_0), (u_+, \xi_+))(P_\mu \tau, t).$$

For notation brevity, we let $\xi = (\xi_-, \xi_0, \xi_+)$.

From now on, u is represented by (ξ, μ) . To prove Proposition 13.5, it is enough to prove the following via the local uniqueness property of the gluing construction.

Proposition 14.2 (Norm-convergence). *Let $\|(\xi, \mu)\|_\varepsilon$ be the Banach norm as defined in (9.2). Then there exists some $0 < \zeta_0 < 1$ and a function $\delta : (0, \varepsilon_0] \rightarrow \mathbb{R}_+$ such that*

$$\|(\xi, \mu)\|_\varepsilon \rightarrow 0$$

uniformly over $u \in \mathcal{M}(K_\varepsilon, J_\varepsilon; z_-, z_+; A_- \# A_+) \cap \bigcup_{0 < \zeta \leq \zeta_0} V_{\zeta, \delta(\varepsilon)}^\varepsilon$ as $\varepsilon \rightarrow 0$.

The rest of this section is devoted to the proof of this norm convergence. We can actually take the function $\delta(\varepsilon)$ to be $\delta(\varepsilon) = \varepsilon$ in this section.

We prove this by contradiction. Suppose that the Proposition is not true, then there exist $0 < \varepsilon_j \leq \varepsilon_0$, $0 < \zeta_j \leq \zeta_0$ and solutions $u^{\varepsilon_j} \in \mathcal{M}(K_{\varepsilon_j}, J_{\varepsilon_j}; z_-, z_+; A_-^j \# A_+^j)$ represented by (ξ_j, μ_j) such that

$$d_{adia}^{\varepsilon_j, \zeta_j}(u^{\varepsilon_j}, (u_-, \chi, u_+)) < \delta(\varepsilon_j) = \varepsilon_j \rightarrow 0,$$

but $\|(\xi_j, \mu_j)\|_{\varepsilon_j} \not\rightarrow 0$. By the assumption u^{ε_j} , u^{ε_j} cannot develop bubbles, so we may assume $u^{\varepsilon_j} \in \mathcal{M}(K_{\varepsilon_j}, J_{\varepsilon_j}; z_-, z_+; A_- \# A_+)$ for a fixed homology class $A_- \# A_+$, and has the uniform C^1 estimate

$$|du^{\varepsilon_j}(\tau, t)| \leq C < \infty. \quad (14.4)$$

Therefore the sequence u^{ε_j} is pre-compact on any given compact interval $[a, b] \times S^1$.

14.1. Exponential map and adiabatic renormalization. In general, on the whole $\mathbb{R} \times S^1$, we define $\xi_j^{app}(\tau, t) \in T_{u_{app}^{\varepsilon_j}(\tau, t)}M$ by

$$u^{\varepsilon_j}(\tau, t) = \exp_{u_{app}^{\varepsilon_j}(\tau, t)} \xi_j^{app}(\tau, t).$$

The exponential map makes sense because of the adiabatic convergence $u^{\varepsilon_j} \rightarrow (u_-, \chi, u_+)$.

For the later purpose, it turns out to be important to use the exponential map at the *genuine solution*

$$u_{glue}^{\varepsilon_j} := \text{Glue}(u_-, \chi, u_+; \varepsilon_j).$$

We also express the same sequence u^{ε_j} as

$$u^{\varepsilon_j}(\tau, t) = \exp_{u_{glue}^{\varepsilon_j}(\tau, t)} \xi_j(\tau, t) \quad (14.5)$$

for $\xi_j(\tau, t) \in T_{u_{glue}^{\varepsilon_j}(\tau, t)}M$. By the triangle inequality, the error estimates (10.2) and the construction of perturbation in implicit function theorem, we have

$$\left\| \xi_{app}^{\varepsilon_j} - \text{Pal}_{app}^{glue}(\varepsilon_j) \left(\xi_{glue}^{\varepsilon_j} \right) \right\|_{\varepsilon_j} \leq \|E(u_{app}^{\varepsilon_j}, u_{glue}^{\varepsilon_j})\|_{\varepsilon_j} \leq CE(\ell) \varepsilon_j^{\frac{1}{p}} \quad (14.6)$$

where

$$E(u_{app}^{\varepsilon_j}, u_{glue}^{\varepsilon_j}) = \exp_{u_{app}^{\varepsilon_j}}^{-1}(u_{glue}^{\varepsilon_j}),$$

(see (14.7)), and $\text{Pal}_{app}^{glue}(\varepsilon_j) : T_{u_{glue}^{\varepsilon_j}(\tau, t)}M \rightarrow T_{u_{app}^{\varepsilon_j}(\tau, t)}M$ is the parallel transport from $u_{glue}^{\varepsilon_j}(\tau, t)$ to $u_{app}^{\varepsilon_j}(\tau, t)$ along the minimal geodesic connecting them. Therefore

$$\begin{aligned} \|\xi_{app}^{\varepsilon_j}\|_{\varepsilon_j} &\leq \left\| \text{Pal}_{app}^{glue}(\varepsilon_j) \left(\xi_{glue}^{\varepsilon_j} \right) \right\|_{\varepsilon_j} + \|E(u_{app}^{\varepsilon_j}, u_{glue}^{\varepsilon_j})\|_{\varepsilon_j} \\ &\leq C \|\xi_{glue}^{\varepsilon_j}\|_{\varepsilon_j} + CE(\ell) \varepsilon_j^{\frac{1}{p}}, \end{aligned}$$

and to prove $\|\xi_{app}^{\varepsilon_j}\|_{\varepsilon_j} \rightarrow 0$, it will be enough to prove

Proposition 14.3. *Let $\xi_{glue}^{\varepsilon_j}$ be as in (14.5). Then*

$$\|\xi_{glue}^{\varepsilon_j}\|_{\varepsilon_j} \rightarrow 0.$$

The remaining section will be occupied by the proof of this proposition.
We consider the exponential map

$$\exp : \mathcal{U} \subset TM \rightarrow M; \quad \exp(x, \xi) := \exp_x(\xi)$$

and denote

$$D_1 \exp(x, \xi) : T_x M \rightarrow T_x M$$

the (covariant) partial derivative with respect to x and

$$d_2 \exp(x, \xi) : T_x M \rightarrow T_x M$$

the usual derivative $d_2 \exp(x, \xi) := T_\xi \exp_x : T_x M \rightarrow T_x M$. We recall the basic property of the exponential map

$$D_1 \exp(x, 0) = d_2 \exp(x, 0) = id.$$

Denote $\chi_j := \text{cm}(u_j)$. Then $\chi_j \rightarrow \chi$ on $[-\ell, \ell]$ by Theorem 4.3. It is also useful to introduce the map

$$E : V_\Delta \rightarrow \mathcal{U}; \quad E(x, y) = \exp_x^{-1}(y) \quad (14.7)$$

where V_Δ is the neighborhood of the diagonal of $M \times M$. Then the following is the standard estimates on this map

$$\begin{aligned} |d_2 \exp(x, v) - \Pi_x^{\exp(x, v)}| &\leq C|v| \\ |D_1 \exp(x, v) - \Pi_x^{\exp(x, v)}| &\leq C|v| \end{aligned} \quad (14.8)$$

for $v \in T_x M$ where C is independent of v , as long as $|v|$ is sufficiently small, say smaller than the injectivity radius of the metric on M . (see [K].)

In the following calculations, to simplify the notation, we *suppress the subindex j from various notations*, i.e. the ε, ξ mean ε_j, ξ_j respectively. We compute

$$\frac{\partial u^\varepsilon}{\partial \tau} = D_1 \exp(u_{glue}^\varepsilon, \xi) \frac{\partial u_{glue}^\varepsilon}{\partial \tau} + d_2 \exp(u_{glue}^\varepsilon, \xi) \frac{\partial \xi}{\partial \tau}.$$

Similarly we compute

$$\frac{\partial u^\varepsilon}{\partial t} = D_1 \exp(u_{glue}^\varepsilon, \xi) \frac{\partial u_{glue}^\varepsilon}{\partial t} + d_2 \exp(u_{glue}^\varepsilon, \xi) \frac{\partial \xi}{\partial t}.$$

We introduce the following invertible linear operators $P(x, v) : T_x M \rightarrow T_x M$ defined by

$$P(x, v) := (d_2 \exp(x, v))^{-1} \circ D_2 \exp(x, v).$$

Then we have the following inequality

Lemma 14.4.

$$|P(x, v) - id| \leq C|v|$$

for a universal constant $C > 0$ depending only on the injectivity radius.

Proof. This is an immediate consequence of (14.8). □

Now we consider the equation

$$0 = \frac{\partial u^\varepsilon}{\partial \tau} + J^\varepsilon \left(\frac{\partial u^\varepsilon}{\partial t} - X_{H^\varepsilon}(u^\varepsilon) \right).$$

We re-write

$$\begin{aligned}
0 &= \left(D_1 \exp(u_{glue}^\varepsilon, \xi) \frac{\partial u_{glue}^\varepsilon}{\partial t} + d_2 \exp(u_{glue}^\varepsilon, \xi) \frac{D\xi}{\partial t} \right) \\
&\quad + J^\varepsilon \left(D_1 \exp(u_{glue}^\varepsilon, \xi) \frac{\partial u_{glue}^\varepsilon}{\partial t} + d_2 \exp(u_{glue}^\varepsilon, \xi) \frac{D\xi}{\partial t} - X_{H^\varepsilon}(u_{glue}^\varepsilon) \right) \\
&= d_2 \exp(u_{glue}^\varepsilon, \xi) \frac{D\xi}{\partial t} + J^\varepsilon \left(d_2 \exp(u_{glue}^\varepsilon, \xi) \frac{D\xi}{\partial t} - DX_{H^\varepsilon}(u_{glue}^\varepsilon)(\xi) \right) \\
&\quad + \left(D_1 \exp(u_{glue}^\varepsilon, \xi) \frac{\partial u_{glue}^\varepsilon}{\partial t} + J^\varepsilon \left(D_1 \exp(u_{glue}^\varepsilon, \xi) \frac{\partial u_{glue}^\varepsilon}{\partial t} - X_{H^\varepsilon}(u_{glue}^\varepsilon) \right) \right) \\
&\quad + J^\varepsilon N(u_{glue}^\varepsilon, \xi) \tag{14.9}
\end{aligned}$$

where $N(u_{glue}^\varepsilon, \xi)$ is the higher order term

$$\begin{aligned}
N(u_{glue}^\varepsilon, \xi) &= X_{H^\varepsilon}(u^\varepsilon) - X_{H^\varepsilon}(u_{glue}^\varepsilon) - DX_{H^\varepsilon}(u_{glue}^\varepsilon)(\xi) \\
&= X_{H^\varepsilon}(\exp(u_{glue}^\varepsilon, \xi)) - X_{H^\varepsilon}(u_{glue}^\varepsilon) - DX_{H^\varepsilon}(u_{glue}^\varepsilon)(\xi) \tag{14.10}
\end{aligned}$$

obtained from the (pointwise) Taylor expansion of $X_{H^\varepsilon}(\exp(x, v))$.

Now we denote the pull-back

$$\widehat{J}^\varepsilon := (d_2 \exp(u_{glue}^\varepsilon, \xi))^{-1} J^\varepsilon d_2 \exp(u_{glue}^\varepsilon, \xi)$$

and

$$\widetilde{J}^\varepsilon := (P(u_{glue}^\varepsilon, \xi))^{-1} J^\varepsilon P(u_{glue}^\varepsilon, \xi).$$

Then we obtain the pointwise inequalities

$$|\widehat{J}^\varepsilon - J^\varepsilon| \leq C|\xi|, \quad |\widetilde{J}^\varepsilon - J^\varepsilon| \leq C|\xi|. \tag{14.11}$$

We also have

$$\begin{aligned}
|(D_1 \exp(u_{glue}^\varepsilon, \xi))^{-1} (X_{K^\varepsilon}(u_{glue}^\varepsilon)) - (X_{K^\varepsilon}(u_{glue}^\varepsilon))| &\leq C_\varepsilon |\xi|, \\
|(d_2 \exp(u_{glue}^\varepsilon, \xi))^{-1} (DX_{K^\varepsilon}(u_{glue}^\varepsilon)) - (DX_{K^\varepsilon}(u_{glue}^\varepsilon))| &\leq C_\varepsilon |\xi|. \tag{14.12}
\end{aligned}$$

With this notation, we can simplify and write (14.9) into

$$\begin{aligned}
&\frac{D\xi}{\partial \tau} + \widehat{J}^\varepsilon \left(\frac{D\xi}{\partial t} - (d_2 \exp(u_{glue}^\varepsilon, \xi))^{-1} (DX_{K^\varepsilon}(u_{glue}^\varepsilon)(\xi)) \right) \\
&= -P(u_{glue}^\varepsilon, \xi) \left(\frac{\partial u_{glue}^\varepsilon}{\partial \tau} + \widetilde{J}^\varepsilon \left(\frac{\partial u_{glue}^\varepsilon}{\partial t} - (D_1 \exp(u_{glue}^\varepsilon, \xi))^{-1} (X_{K^\varepsilon}(u_{glue}^\varepsilon)) \right) \right) \\
&\quad - d_2 \exp(u_{glue}^\varepsilon, \xi)^{-1} (J^\varepsilon N(u_{glue}^\varepsilon, \xi)) \tag{14.13}
\end{aligned}$$

Combining the pointwise inequalities (14.11), (14.12) and the error estimate for u_{glue}^ε , we obtain the differential inequality

$$\left| \frac{D\xi}{\partial t} + \widehat{J}^\varepsilon \left(\frac{D\xi}{\partial t} - (DX_{K^\varepsilon}(u_{glue}^\varepsilon)(\xi)) \right) \right| \leq C(\varepsilon) + C(|\xi|)|\xi|$$

where $C(\varepsilon)$ is the error term for u_{glue}^ε .

For simplicity of exposition, we rewrite this equation (*now with subindex j*) into

$$\begin{aligned}
&\frac{D\xi_j}{\partial \tau} + \widehat{J}^{\varepsilon_j}(\tau, t) \frac{\partial \xi_j}{\partial t} + A_j(u_{glue}^{\varepsilon_j}(\tau, t)) \cdot \xi_j(\tau, t) \\
&= B_j(u_{glue}^{\varepsilon_j}(\tau, t), \xi_j(\tau, t)) + E_{\varepsilon_j}(\tau, t) \tag{14.14}
\end{aligned}$$

where A_j , B_j and E_{ε_j} are defined by

$$\begin{aligned} A_j(x)\xi_j &= -\widehat{J}^{\varepsilon_j} DX_{H^{\varepsilon_j}}(x)(\xi_j), \\ B_j(x, \xi_j) &= -d_2 \exp(u_{glue}^{\varepsilon_j}, \xi)^{-1} (J^{\varepsilon_j} N^{\varepsilon_j}(x, \xi_j)) \\ E_{\varepsilon_j}(\tau, t) &= -P(u_{glue}^{\varepsilon_j}, \xi_j) \left(\frac{\partial u_{glue}^{\varepsilon_j}}{\partial \tau} + \widetilde{J}^{\varepsilon_j} \left(\frac{\partial u_{glue}^{\varepsilon_j}}{\partial t} - (D_1 \exp(u_{glue}^{\varepsilon_j}, \xi))^{-1} (X_{K^{\varepsilon_j}}(u_{glue}^{\varepsilon_j})) \right) \right). \end{aligned}$$

We have

$$|B_j(x, \xi_j)| \leq C |\xi_j|^2 \quad (14.15)$$

for some uniform constant C , provided $\|\xi_j\|_{C^0} \leq$ injectivity radius of the metric g . (Note that this latter requirement is automatic since $\|\xi_j\|_{\varepsilon_j} \rightarrow 0$.) The pointwise inequality (14.15) follows from the inequality (14.8) and the pointwise inequality

$$|X_{H^{\varepsilon}}(\exp_x(v)) - X_{H^{\varepsilon}}(x) - DX_{H^{\varepsilon}}(x)(v)| \leq C|v|^2$$

where C depends only on H , the injectivity radius of the metric as long as $|v| \leq$ injectivity radius of the metric g .

In the transition region $\Omega_{\pm}(\varepsilon_j) := \pm [R(\varepsilon_j), \tau(\varepsilon_j)] \times S^1$, we do not renormalize but consider (14.14) itself. From the adiabatic convergence of $u_j \rightarrow (u_-, \chi, u_+)$ we have

$$|\xi_j(\pm R(\varepsilon_j), t)|, |\xi_j(\pm \tau(\varepsilon_j), t)| \leq \delta_j \quad (14.16)$$

where

$$\delta_j = \varepsilon_j \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

We next consider the region $\Theta(\varepsilon_j) = [-R(\varepsilon_j), R(\varepsilon_j)] \times S^1$ for (τ, t) . We recall that on $\Theta(\varepsilon_j)$ we have $u_{glue}^{\varepsilon_j}(\tau, t) = \chi(\varepsilon_j \tau)$ and so

$$\widehat{J}^{\varepsilon_j}(u_{glue}^{\varepsilon_j}) \equiv J_0 \equiv \widetilde{J}^{\varepsilon_j}(u_{glue}^{\varepsilon_j}).$$

By renormalizing the domain

$$(\overline{\tau}, \overline{t}) = (\varepsilon_j \tau, \varepsilon_j t), \quad \overline{u}_j(\overline{\tau}, \overline{t}) = u_j\left(\frac{\overline{\tau}}{\varepsilon_j}, \frac{\overline{t}}{\varepsilon_j}\right) = u_j(\tau, t),$$

and applying the Hausdorff convergence of $u_j|_{[-R(\varepsilon_j), R(\varepsilon_j)] \times S^1}$ to $\chi|_{[-\ell, \ell]}$, we can write \overline{u}_j

$$\overline{u}_j(\overline{\tau}, \overline{t}) = \exp_{\chi(\overline{\tau})} \overline{\xi}_j(\overline{\tau}, \overline{t}) \quad (14.17)$$

for $(\overline{\tau}, \overline{t}) \in [-\ell, \ell] \times \mathbb{R}/2\pi\varepsilon_j\mathbb{Z}$, a cylinder with radius ε_j . Here we have

$$\overline{\xi}_j(\overline{\tau}, \overline{t}) \in T_{\chi(\overline{\tau})}M.$$

If we restrict (τ, t) on $[-R(\varepsilon_j), R(\varepsilon_j)] \times S^1$, then $\xi_j(\tau, t) = \overline{\xi}_j(\overline{\tau}, \overline{t})$.

Furthermore it easily follows from (14.8) and smoothness of the exponential map that

$$\begin{aligned} & |d_2 \exp(\chi(\overline{\tau}), \overline{\xi}_j(\overline{\tau}, \overline{t}))^{-1} (D_1 \exp(\chi(\overline{\tau}), \overline{\xi}_j(\overline{\tau}, \overline{t}))(\dot{\chi}(\overline{\tau})) - \dot{\chi}(\overline{\tau}))| \\ & \leq C |\overline{\xi}(\tau, t)| |\dot{\chi}(\overline{\tau})| \end{aligned}$$

and

$$|\exp_{\chi(\overline{\tau})}^*(\text{grad } f)(\overline{\xi}_j(\overline{\tau}, \overline{t})) - \text{grad } f(\chi(\overline{\tau}))| \leq C |\overline{\xi}_j(\overline{\tau}, \overline{t})|$$

and

$$|(\exp_{\chi(\overline{\tau})})^* J^{\varepsilon_j}(\chi(\overline{\tau})) - J^{\varepsilon_j}| \leq C |\overline{\xi}_j(\overline{\tau}, \overline{t})|$$

where the constant C depends only on M .

If we take the reparamterization

$$(\bar{\tau}, \bar{t}) = (\varepsilon_j \tau, \varepsilon_j t), \quad \xi_j(\tau, t) = \bar{\xi}_j(\varepsilon_j \tau, \varepsilon_j t),$$

then (14.14) becomes

$$\frac{D\xi_j}{\partial\tau} + J_0(\chi(\varepsilon_j\tau)) \frac{\partial\xi_j}{\partial t} + \varepsilon_j A(\chi(\varepsilon_j\tau)) \cdot \xi_j = \varepsilon_j (B(\chi(\varepsilon_j\tau), \xi_j)) \quad (14.18)$$

where

$$\begin{aligned} A(\chi)\xi &= (\nabla \text{grad } f)(\chi)(\xi) \\ B(\chi, \bar{\xi}_j) &= -d_2 \exp(\chi, \xi_j)^{-1} (J_0 N(\chi, \xi_j)) \\ E_{\varepsilon_j}(\bar{\tau}, \bar{t}) &= -(\bar{\partial}_{J, \varepsilon_j f} u_{glue}^{\varepsilon_j})(\tau, t) = 0. \end{aligned}$$

14.2. Three-interval method of exponential estimates. We first study the equation of ξ_j on the transition regions

$$\Omega_{\pm}(\varepsilon_j) := \pm [R(\varepsilon_j), \tau(\varepsilon_j)] \times S^1.$$

Let $\delta_0 > 0$ be a number much smaller than the injectivity radius of M , and

$$h(\zeta) := \frac{1}{2\pi} |\ln \zeta| > 0.$$

By the adiabatic convergence condition (3), (4), there exists small $0 < \zeta_0 < 1$ and integer j_0 , such that for all $j \geq j_0$ and $\delta_j = \delta(\varepsilon_j)$,

$$\begin{aligned} \text{dist}(u_j(\pm R(\varepsilon_j)), p_{\pm}) &< \delta_j, \\ d_{C^\infty}(\pm[\tau(\varepsilon_j) - h(\zeta_0), \tau(\varepsilon_j) - h(\zeta_0) + 1] \times S^1)(u_j, u_{\pm}^{\varepsilon_j}) &< \delta_j, \\ \text{diam}(u_j(\pm[R(\varepsilon_j), \tau(\varepsilon_j) - h(\zeta_0) + 1] \times S^1)) &< \delta_j. \end{aligned} \quad (14.19)$$

We denote the region

$$\Omega_{\pm\zeta_0}(\varepsilon_j) := \pm [R(\varepsilon_j), \tau(\varepsilon_j) - h(\zeta_0) + 1] \times S^1 \subset \Omega_{\pm}(\varepsilon_j).$$

Without loss of generality we assume $\delta_j \leq \delta_0$. Then for ε_j small, from (14.19) we have

$$u_j(\Omega_{\pm\zeta_0}(\varepsilon_j)) \subset B_{p_{\pm}}(2\delta_0). \quad (14.20)$$

If we identify the neighborhood $B_{p_{\pm}}(2\delta_0)$ into $T_{p_{\pm}}M$ by exponential map and deform the metric and almost complex structure to standard $(g_{p_{\pm}}, J_{p_{\pm}})$, then the ξ_j can be simplified to

$$\xi_j = u_j - u_{glue}^{\varepsilon_j},$$

where the “ $-$ ” is with respect to the linear space structure on $T_{p_{\pm}}M$. Such simplification will not affect the validity of the proof, as explained in section 8 and remark 11.1, for it will only affect the C^1 pointwise estimate of a term of order $C\delta_0$.

We decompose the equation (14.14) into those of 0-mode and higher modes

$$\begin{aligned} \frac{\partial}{\partial\tau}(\xi_j)_0 &= (B)_0(u_{glue}^{\varepsilon_j}(\tau, t), \xi_j) + (E_{\varepsilon_j})_0, \\ \bar{\partial}_J \tilde{\xi}_j &= \tilde{B}(u_{glue}^{\varepsilon_j}(\tau, t), \xi_j) + \widetilde{E_{\varepsilon_j}}. \end{aligned} \quad (14.21)$$

Remark 14.5. In Theorem 1.2 of [MT], the higher mode exponential decay estimate

$$\left| \tilde{\xi}_j \right| \leq C e^{-\sigma(l/\varepsilon_j - |\tau|)} \text{ for } |\tau| \leq l/\varepsilon_j \quad (14.22)$$

has been obtained (Their notation for higher mode is $\phi_0(t, \theta)$ instead of our $\tilde{\xi}_j(\tau, t)$). Their observation was that (14.22) can be reduced to a local L^2 estimate

$$\left\| \tilde{\xi}_j \right\|_{L^2(Z_{III})} \leq \frac{1}{2} \left(\left\| \tilde{\xi}_j \right\|_{L^2(Z_I)} + \left\| \tilde{\xi}_j \right\|_{L^2(Z_{III})} \right) \quad (14.23)$$

on 3 sequential cylinders $Z_I, Z_{II}, Z_{III} \subset [-l/\varepsilon_j, l/\varepsilon_j] \times S^1$ of unit length, namely the cylinders

$$[i-1, i] \times S^1, [i, i+1] \times S^1, [i+1, i+2] \times S^1$$

for some integer i .

To get the best σ in the exponential decay (we need σ to be very close to 2π), we recall in [MT] they defined the constant

$$\gamma(c) = \frac{1}{e^c + e^{-c}}.$$

The importance of $\gamma(c)$ is due to the identity

$$\int_0^1 e^{c\tau} d\tau = \gamma(c) \left[\int_{-1}^0 e^{c\tau} d\tau + \int_1^2 e^{c\tau} d\tau \right] \quad (14.24)$$

which will appear in the L^2 -energy of ξ_j on 3 sequential unit length cylinders later. Notice that when $c > 0$, $\gamma(c)$ is a strictly decreasing function of c .

We recall the elementary

Lemma 14.6 ([MT] Lemma 9.4). *For nonnegative numbers x_k ($k = 0, 1, \dots, N$), if for $1 \leq k \leq N-1$,*

$$x_k \leq \gamma(x_{k-1} + x_{k+1})$$

for some fixed constant $\gamma \in (0, 1/2)$, then for $1 \leq k \leq N-1$,

$$x_k \leq x_0 \xi^{-k} + x_N \xi^{-(N-k)},$$

where $\xi = \frac{1 + \sqrt{1 - 4\gamma^2}}{2\gamma}$.

Remark 14.7. If $\gamma = \gamma(c) = (e^c + e^{-c})^{-1}$, then we can check $\xi = e^c$ and the above inequality becomes the exponential decay estimate

$$x_k \leq x_0 e^{-ck} + x_N e^{-c(N-k)}. \quad (14.25)$$

for $1 \leq k \leq N-1$.

Proposition 14.8. *For any $0 < v < 1$, there exists $N_0 = N_0(v)$ depending on v , such that for all $j > N_0$, on any 3 sequential cylinders Z_I, Z_{II} and Z_{III} in $[-\tau(\varepsilon), \tau(\varepsilon)] \times S^1$ we have*

$$\left\| d\xi_j \right\|_{L^2(Z_{II})} \leq \gamma(4\pi v) \cdot \left(\left\| d\xi_j \right\|_{L^2(Z_I)} + \left\| d\xi_j \right\|_{L^2(Z_{III})} \right) \quad (14.26)$$

and on any 3 sequential cylinders in $[-R(\varepsilon), R(\varepsilon)] \times S^1$ we have

$$\left\| \tilde{\xi}_j \right\|_{L^2(Z_{II})} \leq \gamma(4\pi v) \cdot \left(\left\| \tilde{\xi}_j \right\|_{L^2(Z_I)} + \left\| \tilde{\xi}_j \right\|_{L^2(Z_{III})} \right). \quad (14.27)$$

Proof. We prove (14.26) by contradiction. Suppose that for some sequence $\xi_j \rightarrow 0$ (14.26) is violated on 3 sequential cylinders Z_I^j, Z_{II}^j and Z_{III}^j in $[-\tau(\varepsilon), \tau(\varepsilon)] \times S^1$:

$$\|d\xi_j\|_{L^2(Z_{III}^j)} > \gamma(4\pi v) \left(\|d\xi_j\|_{L^2(Z_I^j)} + \|d\xi_j\|_{L^2(Z_{II}^j)} \right).$$

On $Z_I^j \cup Z_{II}^j \cup Z_{III}^j$ consider the rescaled sequence

$$\widehat{\xi}_j = \xi_j / \|\xi_j\|_{L^\infty(Z_I^j \cup Z_{II}^j \cup Z_{III}^j)},$$

where the denominator $\|\xi_j\|_{L^\infty(Z_I^j \cup Z_{II}^j \cup Z_{III}^j)}$ is never 0, otherwise $\xi_j \equiv 0$ on $Z_I^j \cup Z_{II}^j \cup Z_{III}^j$, contradicting our assumption. Then

$$\begin{aligned} \|\widehat{\xi}_j\|_{L^\infty(Z_I^j \cup Z_{II}^j \cup Z_{III}^j)} &= 1, \\ \|d\widehat{\xi}_j\|_{L^2(Z_{III}^j)} &> \gamma(4\pi v) \cdot \left(\|d\widehat{\xi}_j\|_{L^2(Z_I^j)} + \|d\widehat{\xi}_j\|_{L^2(Z_{II}^j)} \right), \\ \left(\frac{\partial}{\partial \tau} + J_0 \frac{\partial}{\partial t} \right) \widehat{\xi}_j &= (B_j(\xi_j) + E_{\varepsilon_j}) / \|\xi_j\|_{L^\infty(Z_I^j \cup Z_{II}^j \cup Z_{III}^j)}. \end{aligned}$$

We need the following lemma

Lemma 14.9. *For $(\tau, t) \in [-\tau(\varepsilon_j), \tau(\varepsilon_j)] \times S^1$, we have*

$$|E_{\varepsilon_j}(\tau, t)| \leq C |\xi_j(\tau, t)| \left(\left| \frac{\partial u_{glue}^{\varepsilon_j}}{\partial t}(\tau, t) \right| + |X_{K_{\varepsilon_j}}| \right). \quad (14.28)$$

Epecially for $Z_I^j \cup Z_{II}^j \cup Z_{III}^j \subset [-\tau(\varepsilon_j), \tau(\varepsilon_j)] \times S^1$, we have

$$\lim_{j \rightarrow \infty} E_{\varepsilon_j}(\tau, t) / \|\xi_j\|_{L^\infty(Z_I^j \cup Z_{II}^j \cup Z_{III}^j)} = 0.$$

Proof. We recall that $u_{glue}^{\varepsilon_j}$ is a genuine solution for

$$\frac{\partial u}{\partial \tau} + J^{\varepsilon_j} \left(\frac{\partial u}{\partial t} - X_{K^{\varepsilon_j}}(u) \right) = 0. \quad (14.29)$$

Therefore using Lemma 14.4, we obtain

$$|E_{\varepsilon_j}(\tau, t)| \leq C |\widetilde{J}^{\varepsilon_j} - J^{\varepsilon_j}| \left| \frac{\partial u_{glue}^{\varepsilon_j}}{\partial t} \right| + C_1 |(D_1 \exp_{u_{glue}^{\varepsilon_j}})^{-1}(\xi_{\varepsilon_j}) - id| |X_{K_{\varepsilon_j}}(u)|.$$

for some uniform constants C and C_1 . By (14.11), we obtain (14.28). By the definition (3.7) of K_{ε_j} ,

$$K_{\varepsilon_j}(\tau, t, x) = \kappa_0^{\varepsilon_j}(\tau) \cdot \varepsilon_j f(x) \text{ for } |\tau| \leq \tau(\varepsilon_j),$$

so $|X_{K_{\varepsilon_j}}| \rightarrow 0$ as $\varepsilon_j \rightarrow 0$. We also have the $\xi_j := E(u_{app}^{\varepsilon_j}, u_{glue}^{\varepsilon_j})$ satisfying a Cauchy-Riemann equation with an inhomogeneous term of order ε_j , and $|\xi_j|_{C^0} \leq C\varepsilon_j^{1/p}$ by the error estimate (10.2) and Sobolev embedding (13.1). Therefore by the

interior Schauder estimate of ξ_j on $[\tau - \frac{1}{2}, \tau + \frac{1}{2}] \times S^1 \subset [\tau - 1, \tau + 1] \times S^1$ we have $|\frac{\partial}{\partial t} \xi_j(\tau, t)| \leq C\varepsilon_j^{1/p}$. Thus for $|\tau| \leq \tau(\varepsilon_j)$, by the triangle inequality we have

$$\begin{aligned} \left| \frac{\partial u_{glue}^{\varepsilon_j}}{\partial t}(\tau, t) \right| &\leq C \left(\left| \frac{\partial u_{app}^{\varepsilon_j}}{\partial t}(\tau, t) \right| + \left| \frac{\partial}{\partial t} \xi_j(\tau, t) \right| \right) \\ &\leq C (\varepsilon_j + \varepsilon_j^{1/p}) \rightarrow 0, \end{aligned}$$

where in the last inequality we have used the exponential decay of $|du_{\pm}^{\varepsilon_j}|$ for $|\tau| \leq \tau(\varepsilon_j)$. Therefore

$$E_{\varepsilon_j}(\tau, t) / \|\xi_j\|_{L^\infty(Z_I^j \cup Z_{II}^j \cup Z_{III}^j)} \leq C \left(\left| \frac{\partial u_{glue}^{\varepsilon_j}}{\partial t}(\tau, t) \right| + |X_{K^{\varepsilon_j}}| \right) \rightarrow 0$$

as $j \rightarrow \infty$. The lemma follows. \square

Remark 14.10. Here is the place where we need to use the exponential map around the genuine solution $u_{glue}^{\varepsilon_j}$ instead of $u_{app}^{\varepsilon_j}$. If we had used the latter, we would not have the estimate given in this lemma. This is because $u_{app}^{\varepsilon_j}$ is only an approximate solution of (14.29) whose error term

$$\bar{\partial}_{J, K^{\varepsilon_j}}(u_{app}^{\varepsilon_j}) = \frac{\partial u}{\partial \tau} + J^{\varepsilon_j} \left(\frac{\partial u}{\partial t} - X_{K^{\varepsilon_j}}(u_{app}^{\varepsilon_j}) \right),$$

which is not vanishing in general, is hard to compare with $\xi_{app}^{\varepsilon_j}$ when we express $u^{\varepsilon_j} = \exp_{u_{app}^{\varepsilon_j}} \xi_{app}^{\varepsilon_j}$. Then it seems much harder to get

$$E_{\varepsilon_j}(\tau, t) / \|\xi_{app}^{\varepsilon_j}\|_{L^\infty(Z_I^j \cup Z_{II}^j \cup Z_{III}^j)} \rightarrow 0,$$

a key to apply the three-interval method in the following to derive the desired exponential decay of $\xi_{app}^{\varepsilon_j}$. This is the reason why we first replaced $u_{app}^{\varepsilon_j}$ by the genuine solution $u_{glue}^{\varepsilon_j}$ in the exponential map (14.5) in the beginning of our derivation.

Using the lemma and (14.15), we obtain

$$\frac{|B(u_{glue}^{\varepsilon_j}(\tau, t), \xi_j(\tau, t)) + E_{\varepsilon_j}(\tau, t)|}{\|\xi_j\|_{L^\infty(Z_I^j \cup Z_{II}^j \cup Z_{III}^j)}} \leq C \left(|\xi_j| + \left| \frac{\partial u_{glue}^{\varepsilon_j}}{\partial t} \right| \right) \rightarrow 0$$

uniformly over $Z_I^j \cup Z_{II}^j \cup Z_{III}^j \subset [-\tau(\varepsilon_j) - h(\zeta_j), \tau(\varepsilon_j) - h(\zeta_j)] \times S^1$.

After possibly shifting $Z_I^j \cup Z_{II}^j \cup Z_{III}^j$ and taking subsequence of $\widehat{\xi}_j$, we can assume $\widehat{\xi}_j$ C^1 -converges to $\widehat{\xi}_\infty$ on a fixed $Z_I \cup Z_{II} \cup Z_{III}$ (this is guaranteed by C^0 convergence from our adiabatic convergence definition, and elliptic estimate on a length 5 cylinder containing $Z_I^j \cup Z_{II}^j \cup Z_{III}^j$), which satisfies

$$\begin{aligned} \left| \widehat{\xi}_\infty \right|_{L^\infty(Z_I \cup Z_{II} \cup Z_{III})} &= 1, \\ \left\| d\widehat{\xi}_\infty \right\|_{L^2(Z_{II})} &\geq \gamma(4\pi v) \cdot \left(\left\| d\widehat{\xi}_\infty \right\|_{L^2(Z_I)} + \left\| d\widehat{\xi}_\infty \right\|_{L^2(Z_{III})} \right), \\ \left(\frac{\partial}{\partial \tau} + J_0 \frac{\partial}{\partial t} \right) \widehat{\xi}_\infty &= 0. \end{aligned}$$

Then $\widehat{\xi_\infty}$ is a nonzero holomorphic function by the first and third identity. We write $\widehat{\xi_\infty}(\tau, t)$ in Fourier series

$$\widehat{\xi_\infty}(\tau, t) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi k \tau} e^{2\pi k i t} \text{ with } \left\| \widehat{\xi_\infty} \right\|_{L^2(Z_I \cup Z_{II} \cup Z_{III})} \leq 3,$$

where the a_k 's are constant vectors in \mathbb{C}^n . We can explicitly compute

$$\left\| d\widehat{\xi_\infty} \right\|_{L^2([a, b] \times S^1)}^2 = \sum_{k=-\infty}^{\infty} 4\pi^2 k^2 |a_k|^2 \cdot \int_a^b e^{4\pi \tau} d\tau. \quad (14.30)$$

Multiplying (14.24) by $e^{4\pi k}$ and letting $c = 4\pi$ there, we have

$$\int_k^{k+1} e^{4\pi \tau} d\tau = \gamma(4\pi) \left[\int_{k-1}^k e^{4\pi \tau} d\tau + \int_{k+1}^{k+2} e^{4\pi \tau} d\tau \right]. \quad (14.31)$$

By (14.30), (14.31) we see

$$\left\| d\widehat{\xi_\infty} \right\|_{L^2(Z_{III})} = \gamma(4\pi) \cdot \left(\left\| d\widehat{\xi_\infty} \right\|_{L^2(Z_I)} + \left\| d\widehat{\xi_\infty} \right\|_{L^2(Z_{III})} \right).$$

This contradicts with

$$\left\| d\widehat{\xi_\infty} \right\|_{L^2(Z_{III})} \geq \gamma(4\pi v) \cdot \left(\left\| d\widehat{\xi_\infty} \right\|_{L^2(Z_I)} + \left\| d\widehat{\xi_\infty} \right\|_{L^2(Z_{III})} \right)$$

since $\gamma(4\pi) < \gamma(4\pi v)$. The proof of (14.27) is similar. The crucial point is that $\widetilde{\xi}_j$ contains no 0-mode, so the rescaled sequence $\widehat{\xi}_j := \widetilde{\xi}_j / \left\| \widetilde{\xi}_j \right\|_{L^\infty(Z_I^j \cup Z_{II}^j \cup Z_{III}^j)}$ and its limit $\widehat{\xi_\infty}$ are in the higher mode space. Since $\widehat{\xi_\infty}$ is holomorphic, writing it in Fourier series $\widehat{\xi_\infty}(\tau, t) = \sum_{k \neq 0} b_k e^{2\pi k \tau} e^{2\pi k i t}$, we have

$$\left\| \widehat{\xi_\infty} \right\|_{L^2([a, b] \times S^1)}^2 = \sum_{k \neq 0} |b_k|^2 \cdot \int_a^b e^{4\pi \tau} d\tau.$$

This is similar to (14.30), and the remaining steps are the same. We omit the details. \square

Combining the above Lemma and (14.25) we have

Corollary 14.11. *For any $0 < v < 1$, there exists $N_0 = N_0(v)$ depending on v , such that for all $j > N_0$ and $[\tau, \tau + 1] \subset [-\tau(\varepsilon_j), \tau(\varepsilon_j)]$, we have*

$$\int_{[\tau, \tau+1] \times S^1} |d\xi_j|^2 \leq e^{-4\pi v(\tau(\varepsilon_j) - |\tau|)} \left[\int_{[-\tau(\varepsilon_j)-1, -\tau(\varepsilon_j)] \times S^1} |d\xi_j|^2 + \int_{[\tau(\varepsilon_j), \tau(\varepsilon_j)+1] \times S^1} |d\xi_j|^2 \right],$$

and for $[\tau, \tau + 1] \subset [-R(\varepsilon_j), R(\varepsilon_j)]$, we have

$$\int_{[\tau, \tau+1] \times S^1} |\widetilde{\xi}_j|^2 \leq e^{-4\pi v(R(\varepsilon_j) - |\tau|)} \left[\int_{[-R(\varepsilon_j)-1, -R(\varepsilon_j)] \times S^1} |\widetilde{\xi}_j|^2 + \int_{[R(\varepsilon_j), R(\varepsilon_j)+1] \times S^1} |\widetilde{\xi}_j|^2 \right].$$

From these results and standard elliptic estimate on the cylinder $[\tau - \frac{1}{2}, \tau + \frac{1}{2}] \times S^1$, we obtain the following pointwise exponential decay estimate of ξ_j .

Corollary 14.12. *For any $0 < v < 1$, there exists $N_0 = N_0(v)$ depending on v , such that for all $j > N_0$ and $\tau \in [-\tau(\varepsilon_j), \tau(\varepsilon_j)]$, we have*

$$|\nabla \xi_j| \leq C e^{-2\pi v(\tau(\varepsilon_j) - |\tau|)} \left(\|d\xi_j\|_{L^2([-\tau(\varepsilon_j)-1, -\tau(\varepsilon_j)] \times S^1)} + \|d\xi_j\|_{L^2([\tau(\varepsilon_j), \tau(\varepsilon_j)+1] \times S^1)} \right),$$

and for $\tau \in [-R(\varepsilon_j), R(\varepsilon_j)]$, we have

$$|\tilde{\xi}_j| \leq C e^{-2\pi v(R(\varepsilon_j) - |\tau|)} \left(\|\tilde{\xi}_j\|_{L^2([-R(\varepsilon_j)-1, -R(\varepsilon_j)] \times S^1)} + \|\tilde{\xi}_j\|_{L^2([R(\varepsilon_j), R(\varepsilon_j)+1] \times S^1)} \right).$$

The v can be made arbitrarily close to 1.

We fix a v in the above lemma, such that $\frac{p-1}{\delta}v \geq 1$. This is always possible since $\frac{p-1}{\delta} > 1$. We assume $\delta_j \leq \varepsilon_j$ from now on. We estimate ξ_j on 4 regions of $\mathbb{R} \times S^1$.

1. We first study the Banach norm $\|\xi_j\|_{\varepsilon_j}$ on the region $\Omega_{\pm\zeta_0}(\varepsilon_j)$. From the exponential decay estimate we have for $|\tau| \leq \tau(\varepsilon_j) - h(\zeta_0)$ that

$$\begin{aligned} |\nabla \xi_j(\tau, t)| &\leq C e^{2\pi v[\tau - (R(\varepsilon_j) + \frac{p-1}{\delta}S(\varepsilon_j) - h(\zeta_0))]} \|\xi_j\|_{L^2([\tau(\varepsilon_0) - h(\zeta_0), \tau(\varepsilon_0)] \times S^1)} \\ &\leq C e^{2\pi v(\tau - R(\varepsilon_j))} \left(\frac{\varepsilon_j}{l}\right)^{\frac{p-1}{\delta}v} \left(\frac{1}{\zeta_0}\right)^v \delta_j. \end{aligned} \quad (14.32)$$

Integrating $\nabla \xi_j(\tau, t)$ from $R(\varepsilon_j)$ to τ , we have

$$\left| \xi_j(\tau, t) - (\xi_j(R(\varepsilon_j), t))_0 \right| \leq C e^{2\pi v(\tau - R(\varepsilon_j))} \left(\frac{\varepsilon_j}{l}\right)^{\frac{p-1}{\delta}v} \left(\frac{1}{\zeta_0}\right)^v \delta_j. \quad (14.33)$$

Therefore by (14.32) and (14.33), the estimate of $\|\xi_j\|_{\varepsilon_j}$ restricted on the region

$$\left[R(\varepsilon_j), R(\varepsilon_j) + \frac{p-1}{\delta}S(\varepsilon_j) - h(\zeta_0) \right] \times S^1$$

is

$$\begin{aligned} &\int_{R(\varepsilon_j)}^{R(\varepsilon_j) + \frac{p-1}{\delta}S(\varepsilon_j) - h(\zeta_0)} \int_0^1 \left(\left| \xi_j(\tau, t) - (\xi_j(R(\varepsilon_j), t))_0 \right|^p + |\nabla \xi_j(\tau, t)|^p \right) \\ &\quad \cdot e^{2\pi\delta(-\tau + R(\varepsilon_j) + \frac{p-1}{\delta}S(\varepsilon_j))} dt d\tau \\ &\leq C \delta_j^p \left(\frac{1}{\zeta_0}\right)^{vp} \int_0^{\frac{p-1}{\delta}S(\varepsilon_j)} \left(\frac{\varepsilon_j}{l}\right)^{\frac{p-1}{\delta}vp} e^{vps} \cdot e^{-\delta s} \left(\frac{l}{\varepsilon_j}\right)^{p-1} d\tau \quad (\text{where } s = 2\pi(\tau - R(\varepsilon_j))) \\ &\leq C \delta_j^p \left(\frac{1}{\zeta_0}\right)^{vp} \cdot \left(\frac{\varepsilon_j}{l}\right)^{\frac{p-1}{\delta}vp} \left(\frac{l}{\varepsilon_j}\right)^{p-1} \cdot \frac{1}{vp - \delta} e^{(vp - \delta) \cdot 2\pi \frac{p-1}{\delta}S(\varepsilon_j)} \\ &\leq \frac{1}{vp - \delta} C \delta_j^p \left(\frac{1}{\zeta_0}\right)^{vp} \cdot \left(\frac{\varepsilon_j}{l}\right)^{\frac{p-1}{\delta}vp - (p-1) - \frac{(vp - \delta)(p-1)}{\delta}} \\ &= \frac{1}{vp - \delta} C \delta_j^p \left(\frac{1}{\zeta_0}\right)^{vp} \cdot 1 \rightarrow 0. \end{aligned}$$

2. We consider the regions

$$\begin{aligned} \Phi_{\zeta_0}(\varepsilon_j) &: = [-\tau(\varepsilon_j) - h(\zeta_0), \tau(\varepsilon_j) - h(\zeta_0)] \times S^1. \\ \Phi_{\zeta_0}^+(\varepsilon_j) &: = [-\tau(\varepsilon_j) - h(\zeta_0) + 1, \tau(\varepsilon_j) - h(\zeta_0) + 1] \times S^1. \end{aligned}$$

From the interior Schauder estimate we have

$$\begin{aligned}
& \|\nabla \xi_j\|_{C^0(\Phi_{\zeta_0}(\varepsilon_j))} \\
& \leq \|\xi_j\|_{C^{1,\alpha}(\Phi_{\zeta_0}(\varepsilon_j))} \\
& \leq C \left(\left\| J^{\varepsilon_j}(\chi(\varepsilon_j \tau)) \frac{\partial \xi_j}{\partial t} + \varepsilon_j A_j(\chi(\varepsilon_j \tau)) \cdot \xi_j \right\|_{C^\alpha(\Phi_{\zeta_0}^+(\varepsilon_j))} + \|\xi_j\|_{C^0(\Phi_{\zeta_0}^+(\varepsilon_j))} \right) \\
& \leq C \left(\|B_j(\chi(\varepsilon_j \tau), \xi_j)\|_{C^0(\Phi_{\zeta_0}^+(\varepsilon_j))} + \|E_{\varepsilon_j}(\tau, t)\|_{C^0(\Phi_{\zeta_0}^+(\varepsilon_j))} + \|\xi_j\|_{C^0(\Phi_{\zeta_0}^+(\varepsilon_j))} \right) \\
& \leq C \left(\|\nabla \xi_j\|_{C^0(\Phi_{\zeta_0}^+(\varepsilon_j))} \|\xi_j\|_{C^0(\Phi_{\zeta_0}^+(\varepsilon_j))} + \|\xi_j\|_{C^0(\Phi_{\zeta_0}^+(\varepsilon_j))}^2 \right. \\
& \quad \left. + \|\xi_j\|_{C^0(\Phi_{\zeta_0}^+(\varepsilon_j))} \right), \tag{14.34}
\end{aligned}$$

where the last inequality is because B is quadratic, and $\|E_{\varepsilon_j}(\tau, t)\|_{C^0(\Phi_{\zeta_0}^+(\varepsilon_j))} \leq C \|\xi_j\|_{C^0(\Phi_{\zeta_0}^+(\varepsilon_j))}$ from (14.28). Using the C^∞ uniform convergence of ξ_j outside $\Phi_{\zeta_0}(\varepsilon_j)$, we have

$$\|\nabla \xi_j\|_{C^0(\Phi_{\zeta_0}^+(\varepsilon_j))} \leq \|\nabla \xi_j\|_{C^0(\Phi_{\zeta_0}(\varepsilon_j))} + \delta_j.$$

Plugging in (14.34) and noting $\|\xi_j\|_{C^0(\Phi_{\zeta_0}^+(\varepsilon_j))} \leq \delta_j$, we have

$$(1 - C\delta_j) \|\nabla \xi_j\|_{C^0(\Phi_{\zeta_0}(\varepsilon_j))} \leq C(\delta_j^2 + \delta_j^2 + \delta_j),$$

so for $\delta_j < \min\{\frac{1}{2C}, 1\}$ we have

$$\|\nabla \xi_j\|_{C^0(\Phi_{\zeta_0}(\varepsilon_j))} \leq 2C(3\delta_j) = 6C\delta_j.$$

3. Then we study the equation of ξ_j on the region

$$\Theta(\varepsilon_j) := [-R(\varepsilon_j), R(\varepsilon_j)] \times S^1.$$

For the higher mode $\tilde{\xi}_j$, by (14.32) we have

$$\left| \nabla \tilde{\xi}_j(\tau, t) \right| \leq \left| \nabla \xi_j(\tau, t) \right| \leq C\delta_j \left(\frac{1}{\zeta_0} \right)^v \left(\frac{\varepsilon_j}{l} \right)^{\frac{p-1}{\delta}v} e^{2\pi v(\tau - R(\varepsilon_j))} \leq C\delta_j \left(\frac{1}{\zeta_0} \right)^v \varepsilon_j. \tag{14.35}$$

where in the last inequality we have used that $\frac{p-1}{\delta}v > 1$ and $|\tau| \leq R(\varepsilon_j)$.

We notice that on $\Theta(\varepsilon) = [-R(\varepsilon), R(\varepsilon)] \times S^1$, the weighting function ε^{1-p} dominates the power weight $\|\cdot\|_{W_{\rho_\varepsilon}^{1,p}}$, up to constant factor $(2l)^\delta$, because

$$\rho_\varepsilon(\tau) = \varepsilon^{1-p+\delta}(1 + |\tau|)^\delta \leq \varepsilon^{1-p+\delta}(2l/\varepsilon)^\delta = (2l)^\delta \varepsilon^{1-p}.$$

So for higher mode $\tilde{\xi}_j$ we obtain

$$\begin{aligned} \left\| \tilde{\xi}_j \right\|_{W_{\beta\delta, \varepsilon_j}^{1,p}(\Theta(\varepsilon_j))}^p &\leq \int_{-R(\varepsilon_j)}^{R(\varepsilon_j)} \int_0^1 \left(\left| \tilde{\xi}_j \right|^p + |\nabla \tilde{\xi}_j|^p \right) (2l)^\delta \varepsilon_j^{1-p} dt d\tau \\ &\leq (2l)^\delta \int_{-R(\varepsilon_j)}^{R(\varepsilon_j)} 2 \left(C\delta_j \left(\frac{1}{\zeta_0} \right)^v \varepsilon_j \right)^p \varepsilon_j^{1-p} d\tau \\ &= 2C^p (2l)^\delta \left(\frac{1}{\zeta_0} \right)^{vp} \delta_j^p \rightarrow 0. \end{aligned}$$

For the 0-mode $(\xi_j)_0$, noticing that for $|\tau| \leq l/\varepsilon$ the error term is 0, from the equation of $(\xi)_0$ we have

$$\begin{aligned} \left| \nabla (\xi_j)_0(\tau) \right| &= \left| \frac{\partial}{\partial \tau} (\xi_j)_0 \right| = \left| \varepsilon_j A(\chi(\varepsilon_j \tau)) (\xi_j)_0 + (B(\xi_j))_0 \right| \\ &\leq C(\varepsilon_j \delta_j + \delta_j^2) \quad (\because B(\xi_j) \text{ is quadratic}) \\ &\leq 2C\varepsilon_j \delta_j \quad (\because \delta_j \leq \varepsilon_j). \end{aligned} \tag{14.36}$$

Therefore

$$\begin{aligned} \left\| (\xi_j)_0 \right\|_{W_{\varepsilon_j}^{1,p}(\Theta(\varepsilon_j))}^p &= \int_{-R(\varepsilon_j)}^{R(\varepsilon_j)} \left(\varepsilon_j \left| (\xi_j)_0 \right|^p + \varepsilon_j^{1-p} |\nabla (\xi_j)_0|^p \right) d\tau \\ &\leq \int_{-R(\varepsilon_j)}^{R(\varepsilon_j)} \left(\varepsilon_j \delta_j^p + \varepsilon_j^{1-p} (2C\varepsilon_j \delta_j)^p \right) d\tau \\ &\leq C(l\delta_j^p + R(\varepsilon_j)\varepsilon_j \delta_j^p) = Cl\delta_j^p \rightarrow 0. \end{aligned}$$

where the second inequality is by (14.36).

By Sobolev embedding $\left| (\xi_j(\pm l/\varepsilon_j))_0 \right| \leq C \left\| (\xi_j)_0 \right\|_{W_{\varepsilon_j}^{1,p}(\Theta(\varepsilon_j))} \leq C\delta_j$.

Combining these we have

$$\begin{aligned} \left\| \xi_j \right\|_{\varepsilon_j \mid \Theta(\varepsilon_j) \cup \Omega_{\pm \zeta_0}(\varepsilon_j)} &= \left\| \tilde{\xi}_j \right\|_{W_{\beta\delta, \varepsilon_j}^{1,p}(\Theta(\varepsilon_j) \cup \Omega_{\pm \zeta_0}(\varepsilon_j))} + \left\| (\xi_j)_0 \right\|_{W_{\varepsilon_j}^{1,p}(\Theta(\varepsilon_j) \cup \Omega_{\pm \zeta_0}(\varepsilon_j))} \\ &\quad + \left| (\xi_j(\pm l/\varepsilon_j))_0 \right| \\ &\leq C\delta_j \rightarrow 0, \end{aligned}$$

where the constant C is uniform for all $0 < \varepsilon_j \leq \varepsilon_0$ and $l \geq l_0$.

4. Outside the region $\Theta(\varepsilon_j) \cup \Omega_{\pm \zeta_0}(\varepsilon_j)$, namely for $|\tau| > \tau(\varepsilon_j) - h(\zeta_0)$, by the C^∞ uniform convergence of u_j to $u_\pm^{\varepsilon_j}$ it is easy to see

$$\left\| \xi_j \right\|_{\varepsilon_j \mid \Sigma_{\varepsilon_j} \setminus (\Theta(\varepsilon_j) \cup \Omega_{\pm \zeta_0}(\varepsilon_j))} = \left\| \xi_j \right\|_{\varepsilon_j \mid \Sigma_{\pm} \setminus U_{\pm}(\zeta_0)} \rightarrow 0.$$

This finishes the proof

$$\left\| \xi_j \right\|_{\varepsilon_j} \rightarrow 0$$

and so the proof of Proposition 13.5.

15. VARIANTS OF ADIABATIC GLUING

In this section, we discuss various cases to which similar adiabatic gluing construction can be applied. Since the necessary analysis will be small modifications of the current constructions, we will be brief in our discussion.

15.1. Pearl complex in Hamiltonian case. Let $f : M \rightarrow \mathbb{R}$ be a Morse function. The the Floer equation for the Hamiltonian εf is

$$\frac{\partial u}{\partial \tau} + J(u) \left(\frac{\partial u}{\partial t} - \varepsilon X_f(u) \right) = 0 \quad (15.1)$$

for $u : \mathbb{R} \times S^1 \rightarrow M$ with asymptote $u(\pm\infty, t) = z_{\pm}(t)$, where $z_{\pm}(t)$ are Hamiltonian 1-periodic orbits of εf . In the papers [Oh3], [Oh4], the first named author studied the adiabatic degeneration of the moduli space of solutions satisfying the above equation as $\varepsilon \rightarrow 0$. [MT] studied similar adiabatic degeneration for twisted holomorphic sections in Hamiltonian S^1 -manifolds. The limiting moduli space as $\varepsilon = 0$ consists of sphere-flow-sphere configurations which Biran and Cornea call “pearl complexes” and is defined as the following

Definition 15.1. The configuration

$$u := (p, \chi_{-\infty}, u_1, \chi_1, u_2, \chi_2, \dots, u_k, \chi_{\infty}, q)$$

is called a pearl configuration if $u_i : S^2 \cong \mathbb{R} \times S^1 \rightarrow M$ are J -holomorphic spheres with marked points $u_i(o_{\pm})$ where $o_{\pm} = \{\pm\infty\} \times S^1$, and each $\chi_i : [-l_i, l_i] \rightarrow M$ is a gradient segment of the Morse function f connecting $u_i(o_+)$ to $u_{i+1}(o_-)$, $\chi_{-\infty}$ connecting the critical point p to $u_1(o_-)$ and χ_{∞} connecting $u_k(o_+)$ to the critical point q .

We define the moduli space

$$\begin{aligned} \mathcal{M}_2(M, J; A_i) &= \left\{ (u_i, o_{\pm}) \mid u_i : S^2 \cong \mathbb{R} \times S^1 \rightarrow M, o_{\pm} \in S^2, \right. \\ &\quad \left. \bar{\partial}_J u_i = 0, [u_i] = A_i \in H_2(M, \mathbb{Z}) \right\} / \mathbb{R} \times S^1 \end{aligned}$$

where the last $\mathbb{R} \times S^1$ is the automorphism group \mathbb{R} -translation and S^1 rotation, and the evaluation maps

$$ev_{\pm}^i : \mathcal{M}_2(M, J; A_i) \rightarrow M, \quad u_i \rightarrow u_i(o_{\pm}).$$

Consider the map

$$id \times \left(\Pi_{i=0}^{k-1} \phi_f^{2l_i} \circ ev_+^i \times ev_-^{i+1} \right) \times id \quad : \quad (15.2)$$

$$\begin{aligned} W^u(p) \times \Pi_{i=1}^k \mathcal{M}_2(M, J; A_i) \times W^s(q) &\rightarrow \Pi_{i=1}^{k+1}(M \times M), \\ (x, u_1, \dots, u_k, y) &\rightarrow \left(x, \Pi_{i=1}^{k-1} \left(\phi_f^{2l_i} u_i(o_+), u_{i+1}(o_-) \right), y \right), \end{aligned}$$

where ϕ_f^{2l} is the time- $2l$ flow of the Morse function f , $W^u(p)$ and $W^s(q)$ are unstable and stable manifolds of p and q respectively.

Definition 15.2. The moduli space of pearl configuration with flow length vector $\vec{l} := (l_1, l_2, \dots, l_k)$ connecting p to q with the J -holomorphic spheres u_i ($i = 1, 2, \dots, k$) in homology class $\vec{A} = (A_1, A_2, \dots, A_k)$ is defined to be

$$\mathcal{M}_{\text{pearl}}^{\vec{l}}(p, q; f; \vec{A}) = \left(id \times \left(\Pi_{i=0}^{k-1} \phi_f^{2l_i} \circ ev_+^i \times ev_-^{i+1} \right) \times id \right)^{-1} (\Pi_{i=1}^{k+1} \Delta),$$

where $\Delta \subset M \times M$ is the diagonal.

We can give the obvious $W^{1,p}$ Banach manifold $\mathcal{B}_{\text{pearl}}^{\vec{l}}(p, q)$ to host $\mathcal{M}_{\text{pearl}}^{\vec{l}}(p, q; f; \vec{A})$, and a natural section e of the Banach bundle $\mathcal{L}_{\text{pearl}}^{\vec{l}}(p, q) \rightarrow \mathcal{B}_{\text{pearl}}^{\vec{l}}(p, q)$,

$$e : \mathcal{B}_{\text{pearl}}^{\vec{l}}(p, q) \rightarrow \mathcal{L}_{\text{pearl}}^{\vec{l}}(p, q),$$

such that

$$\mathcal{M}_{\text{pearl}}^{\vec{l}}(p, q; f; \vec{A}) = e^{-1}(0).$$

We let the linearization of e at $u \in \mathcal{M}_{\text{pearl}}^{\vec{l}}(p, q; f; \vec{A})$ to be $E(u)$.

We assume the pearl configuration $u := (p, \chi_{-\infty}, u_1, \chi_1, u_2, \chi_2, \dots, u_k, \chi_k, q)$ satisfies the “sphere-flow-sphere” transversality defined as the following, which was also defined in [BC] for the case with Lagrangian boundary condition.

Definition 15.3. The pearl configuration $u = (p, \chi_{-\infty}, u_1, \chi_1, u_2, \chi_2, \dots, u_k, \chi_k, q)$ satisfies the “sphere-flow-sphere” transversality if the map

$$id \times \left(\prod_{i=0}^{k-1} \phi_f^{2l_i} \circ ev_+^i \times ev_-^{i+1} \right) \times id$$

in (15.2) is transversal to the diagonal $\Pi_{i=1}^{k+1} \Delta \subset \Pi_{i=1}^{k+1}(M \times M)$.

The “sphere-flow-sphere” transversality is known to be achievable for generic J and f (see [BC] for example). Small modifications of Proposition 6.2, Corollary 6.3 and Proposition 6.4 leads to the corresponding

Proposition 15.4. For any $u \in \mathcal{M}_{\text{pearl}}^{\vec{l}}(p, q; f; \vec{A})$, the operator $E(u)$ is a Fredholm operator and

$$\text{Index } E(u) = \mu_f(p) - \mu_f(q) + \sum_{i=1}^k 2c_1(A_i). \quad (15.3)$$

where μ_f is the Morse index for critical points of f .

Corollary 15.5. Suppose that each $u_i \in \mathcal{M}_2(M, J; A_i)$ is Fredholm regular, and each gradient segment $\chi_{\pm\infty}$ belongs to a full gradient trajectory that is Fredholm regular, then $u \in \mathcal{M}_{\text{pearl}}^{\vec{l}}(p, q; f; \vec{A})$ is Fredholm regular (in the sense that $E(u)$ is surjective) if and only if the configuration u satisfies the “sphere-flow-sphere” transversality in definition 15.3.

Proposition 15.6. Suppose that each $u_i \in \mathcal{M}_2(M, J; A_i)$ is Fredholm regular. Then there exists a dense subset of $f \in C^\infty(M)$ such that any element u in

$$\mathcal{M}_{\text{pearl}}^{\text{para}}(p, q; f; \vec{A}) = \bigcup_{l_i > 0} \mathcal{M}_{\text{pearl}}^{\vec{l}}(p, q; f; \vec{A})$$

is Fredholm regular, in the sense that $E(u)$ is surjective. Therefore $\mathcal{M}_{\text{pearl}}^{\text{para}}(p, q; f; \vec{A})$ is a smooth manifold with dimension equal to the index of $E(u)$:

$$\dim \mathcal{M}_{\text{pearl}}^{\text{para}}(p, q; f; \vec{A}) = \text{Index } E(u) = \mu_f(p) - \mu_f(q) + \sum_{i=1}^k 2c_1(A_i) + 1.$$

The gluing problem from “pearl configuration” to nearby Floer trajectories was mentioned and left as a future work in [Oh1], [MT]. Now we can study the gluing from the pearl configuration $u = (p, \chi_{-\infty}, u_1, \chi_1, u_2, \chi_2, \dots, u_k, \chi_k, q)$ to nearby Floer trajectories satisfying (15.1), using the techniques from this paper. We assume J and f are generic such that $E(u)$ in Proposition 15.6 is surjective. The outcome is

Theorem 15.7. *For any pearl configuration $u = (p, \chi_{-\infty}, u_1, \chi_1, u_2, \chi_2, \dots, u_k, \chi_\infty, q)$ in $\mathcal{M}_{\text{pearl}}^{\text{para}}(p, q; f; \vec{A})$ whose $E(u)$ is surjective, then for sufficiently small $\varepsilon > 0$, nearby u we have the solutions $u_{\vec{\theta}, \vec{d}}^\varepsilon(\tau, t)$ of Floer equation (15.1) parameterized by the gluing parameters $\varepsilon > 0$, $\vec{\theta} = (\theta_1, \theta_2, \dots, \theta_k) \in (S^1)^k$ and $\vec{d} = (d_1, d_2, \dots, d_k) \in (\mathbb{R})^k$, with the asymptotes $u_{\vec{\theta}, \vec{d}}^\varepsilon(-\infty, t) = p$, and $u_{\vec{\theta}, \vec{d}}^\varepsilon(+\infty, t) = q$.*

Proof. (Outline) Locally the gluing is the same, in the sense that it is the gluing of gradient segments (with noncritical joint points) with J -holomorphic curves. The construction of approximate solution is the same, except that for any J -holomorphic sphere u_i we have the $S^1 \times \mathbb{R}$ family of J -holomorphic spheres $u_i(\tau + d_i, \theta + \theta_i)$ in pregluing. The $\bar{\partial}$ -error estimate is the same, with the pair $(K_\varepsilon, J_\varepsilon)$ in the piece (3) in section 10 replaced by $(\varepsilon f, J_0)$ hence $\bar{\partial}_{(K_\varepsilon, J_\varepsilon)}$ replaced by $\bar{\partial}_{(\varepsilon f, J_0)}$. The weight $\beta_{\delta, \varepsilon}(\tau)$ on $\chi_{\pm\infty}$ is not of a polynomial weight but of an exponential weight when $\chi_{\pm\infty}(\tau)$ approaches the critical points p, q , but by remark 10.2 this does not destroy $\bar{\partial}_{J, \varepsilon f}$ -error estimate. The construction and estimates of the right inverse are the same. The quadratic estimate remains the same, where the domain Riemann surface $\Sigma_\varepsilon \simeq \mathbb{R} \times S^1$ is equipped with standard measure on disjoint cylinders $[-l_i/\varepsilon - \frac{p-1}{\delta}S(\varepsilon), l_i/\varepsilon + \frac{p-1}{\delta}S(\varepsilon)] \times S^1$ and $(\pm\infty, 0] \times S^1$, glued with standard middle annulus of S^2 (with compact measure). \square

Notice that we do not have the surjectivity part in this theorem, because a sequence of solutions u^ε of Floer equation (15.1) may develop multiple covering J -holomorphic spheres or multiple covering of gradient segments in the limit, which lacks Fredholm regularity; The bubbling sphere may also occur at any interior point on the gradient segment χ , or worse, the joint points. For that situation we have not fully developed the gluing analysis.

15.2. Pearl complex in Lagrangian case. Let (M, ω) be a compact symplectic manifold with compatible almost complex structure J and L be a compact Lagrangian submanifold. For a Morse function $f : L \rightarrow \mathbb{R}$, we extend f to a Morse function $f : M \rightarrow \mathbb{R}$ such that in a Weinstein neighborhood of L which is symplectomorphic to T^*M , f is constant along each fiber. Let $\phi_{\varepsilon f}^t$ be the time- t Hamiltonian flow of f . Then $L_{\varepsilon f} := \phi_{\varepsilon f}^1 L$ is a Lagrangian submanifold Hamiltonian isotopic to L . For generic f , L and $L_{\varepsilon f}$ transversally intersect. Similar to the Hamiltonian case, we study the J -holomorphic stripe $u : \mathbb{R} \times [0, 1] \rightarrow M$ satisfying

$$\begin{aligned} \frac{\partial u}{\partial \tau} + J(u) \frac{\partial u}{\partial t} &= 0, \quad u(\mathbb{R}, 0) \in L \text{ and } u(\mathbb{R}, 1) \in L_{\varepsilon f} \\ u(-\infty, \cdot) &= p, \quad u(+\infty, \cdot) = q \end{aligned} \quad (15.4)$$

where p, q are intersections of L and $L_{\varepsilon f}$. The limiting moduli space as $\varepsilon \rightarrow 0$ consists of “pearl complexes” (or disk-flow-disk configurations) defined as the following

Definition 15.8. The configuration

$$u := (p, \chi_{-\infty}, u_1, \chi_1, u_2, \chi_2, \dots, u_k, \chi_\infty, q)$$

is called a pearl configuration if $u_i : D^2 \setminus \{\pm 1\} \cong \mathbb{R} \times [0, 1] \rightarrow M$ are J -holomorphic discs with lower and upper boundaries ending on L and $L_{\varepsilon f}$ respectively, with marked points $u_i(o_\pm)$ where $o_\pm = \{\pm 1\} \in D^2 = \{\pm\infty\} \times [0, 1] \in \mathbb{R} \times [0, 1]$, and each

$\chi_i : [-l_i, l_i] \rightarrow L$ is a gradient segment of the Morse function f connecting $u_i(o_+)$ to $u_{i+1}(o_-)$, $\chi_{-\infty}$ connecting the critical point p to $u_1(o_-)$ and χ_{∞} connecting $u_k(o_+)$ to the critical point q .

Suppose that each u_i is Fredholm regular. The disk-flow-disk transversality has been defined in [BC] and proven to be achievable for generic (J, f) . Parallel to the previous section, we can define the pearl complex moduli space $\mathcal{M}_{\text{pearl}}^{\text{para}}(p, q; L, f; \vec{A})$, establish the index formula for u in such moduli space, and show $E(u)$ is surjective for generic (J, f) . These are more or less standard so we omit details (See [BC] for precise definitions of pearl complex moduli space and index formula). The following is the gluing theorem.

Theorem 15.9. *For any pearl configuration $u = (p, \chi_{-\infty}, u_1, \chi_1, u_2, \chi_2, \dots, u_k, \chi_{\infty}, q)$ in $\mathcal{M}_{\text{pearl}}^{\text{para}}(p, q; L, f; \vec{A})$ whose $E(u)$ is surjective, then for sufficiently small $\varepsilon > 0$, nearby u we have J -holomorphic stripes $u_{\vec{d}}^{\varepsilon}(\tau, t)$ of equation (15.4) parameterized by the gluing parameters $\varepsilon > 0$ and $\vec{d} = (d_1, d_2, \dots, d_k) \in (\mathbb{R})^k$, with the asymptotes $u_{\vec{d}}^{\varepsilon}(-\infty, t) = p$, and $u_{\vec{d}}^{\varepsilon}(+\infty, t) = q$.*

Proof. (Outline) The equation (15.4) can be changed into the following form which is very similar to the Hamiltonian case: For any $u : \mathbb{R} \times [0, 1] \rightarrow M$, let $u^{\varepsilon}(\tau, t) := \phi_{\varepsilon f}^{-t} \circ u(\tau, t)$. Then u satisfies (15.4) if and only if u^{ε} satisfies

$$\begin{aligned} \frac{\partial u^{\varepsilon}}{\partial \tau} + J_t^{\varepsilon f}(u^{\varepsilon}) \frac{\partial u^{\varepsilon}}{\partial t} + \varepsilon \nabla f(u^{\varepsilon}) &= 0, \quad u^{\varepsilon}(\mathbb{R}, 0) \text{ and } u^{\varepsilon}(\mathbb{R}, 1) \subset L, \\ u^{\varepsilon}(-\infty, \cdot) &= p \quad \text{and} \quad u^{\varepsilon}(+\infty, \cdot) = q. \end{aligned} \quad (15.5)$$

where $\{J_t^{\varepsilon f}\}_{0 \leq t \leq 1} := \{(\phi_{\varepsilon f}^t)_* J(\phi_{\varepsilon f}^{-t})_*\}_{0 \leq t \leq 1}$ is a 1-parameter family of compatible almost complex structures.

From the pearl configuration u we can construct the approximate solution for equation (15.5) from J -holomorphic discs u_i and gradient segments χ_i ($i = 1, 2, \dots, k$) and $\chi_{\pm\infty}$ as in the Hamiltonian case, but there is one difference: The almost complex structure $J_t^{\varepsilon f}$ is t -dependent.

The t -dependent $J_t^{\varepsilon f}$ seems to destroy the decomposition of 0-mode and higher modes of variation vector fields on χ as in the Hamiltonian case, but actually here we have even better situation: the linearized operator of $\bar{\partial}_{J_t^{\varepsilon f}, \varepsilon f}$ is canonically related to linearized operator of $\frac{d}{d\tau} - \text{grad}(\varepsilon f)$, and the right inverse bound for $D_{\chi_{\varepsilon}} \bar{\partial}_{J_t^{\varepsilon f}, \varepsilon f}$ can be controlled by the right inverse bound of $\frac{D}{d\tau} - \nabla \text{grad}(\varepsilon f)$ up to a uniform constant (as in the Proposition 6.1 of [FO1] and Proposition 4.6 in [Oh2]). Therefore we do not need to decompose the 0-mode and higher modes of variation vector fields on χ in the Banach manifold setting, and the weighting function $\beta_{\varepsilon, \delta}(\tau)$ on the portion over $\chi_i(\varepsilon\tau)$ ($-l_i/\varepsilon \leq \tau \leq l_i/\varepsilon$) is just the ε -adiabatic weight, namely

$$\|\xi\|_{W_{\varepsilon}^{1,p}[-l_i/\varepsilon, l_i/\varepsilon] \times [0,1]}^p = \int_0^1 \int_{-l/\varepsilon}^{l/\varepsilon} (\varepsilon^2 |\xi_0|^p + \varepsilon^{2-p} |\nabla \xi_0|^p) d\tau dt$$

as in [FO1]. The weighting function $\beta_{\varepsilon, \delta}(\tau)$ on the portion over $u_i^{\varepsilon}(\tau)$ remains the same, namely $\beta_{\varepsilon, \delta}(\tau) = e^{\delta|\tau|}$ for large $|\tau|$ of the ends $u_i^{\varepsilon}(\tau)$. The construction of combined right inverse and uniform quadratic estimates are similar to the Hamiltonian case. \square

The limit of moduli spaces of J -holomorphic $(k+1)$ -gons ending on $(L_{\varepsilon f_0}, \dots, L_{\varepsilon f_k})$ as $\varepsilon \rightarrow 0$ consists of “cluster complexes”, which are J -holomorphic discs u_i ending on L connected with *gradient flow trees* χ_j on L . In [FO1], the gluing from gradient flow trees χ_j to “thin” J -holomorphic polygons was known. The gluing of “thin” J -holomorphic polygon with “thick” J -holomorphic discs u_i locally is the same as the case of pearl complex as above.

Again, we do not have the surjectivity part in the gluing theorem. The reasons are similar to Hamiltonian case. The multicovering is the essential difficulty, and we also need the joint point to be immersed to prove surjectivity. However, for pearl complex moduli spaces of virtual dimension 0, 1 for monotone Lagrangian submanifolds, in Proposition 3.13 of [BC] it has been proved that the J -holomorphic discs in pearl complex are Fredholm regular, simple (non-multicovering) and with absolutely distinct, provided (J, f) is generic. Then, for simple J -holomorphic discs, relative version (with Lagrangian boundary condition) of Theorem 2.6 in [OZ2] implies that the condition of existence of non-immersed point on some J -holomorphic disk of the pearl complex cut down the dimension of pearl complex moduli space by $\dim M - 2$, provided (J, f) is generic. So with non-immersed condition the moduli space will have negative dimension if $\dim M \geq 4$. When $\dim M = 2$, M is a Riemann surface, and the J -holomorphic discs are just discs on surfaces M with embedded boundary curve L . If the J -holomorphic disc is simple, by Riemann mapping theorem it is immersed. Hence we have

Corollary 15.10. *Let (M, ω) be monotone and $L \subset (M, \omega)$ a monotone Lagrangian submanifold. Then for generic (J, f) , all J -holomorphic discs in pearl complex moduli spaces of virtual dimension 0, 1, are simple, Fredholm regular and immersed.*

Now with immersion condition, similar analysis as Hamiltonian case can establish surjectivity so we have

Theorem 15.11. *Let (M, ω) be a monotone symplectic manifold and $L \subset (M, \omega)$ be a monotone Lagrangian submanifold. Fix a Darboux neighborhood U of L and identify U with a neighborhood of the zero section in T^*L . Consider $k+1$ Hamiltonian deformations of L by autonomous Hamiltonian functions $F_0, \dots, F_k : M \rightarrow \mathbb{R}$ such that*

$$F_i = \chi f_i \circ \pi$$

where $f_0, f_1, \dots, f_k : L \rightarrow \mathbb{R}$ are generic Morse functions, χ is a cut-off function such that $\chi = 1$ on U and supported nearby U . Now consider

$$L_{i,\varepsilon} = \text{Graph}(\varepsilon df_i) \subset U \subset M, \quad i = 0, \dots, k.$$

Assume transversality of L_i 's of the type given in [FO1]. Consider the intersections $p_i \in L_i \cap L_{i+1}$ such that

$$\dim \mathcal{M}(L_0, \dots, L_k; p_0, \dots, p_k) = 0, 1.$$

Then when ε is sufficiently small, the moduli space $\mathcal{M}(L_{0,\varepsilon}, \dots, L_{k,\varepsilon}; p_0, \dots, p_k)$ is diffeomorphic to the moduli space of pearl complex defined in [BC].

16. DISCUSSION AND AN EXAMPLE

16.1. The immersion condition: an example of adiabatic limit in \mathbb{CP}^n .
 Consider the standard \mathbb{CP}^n equipped with Fubini-Study metric g_{FS} . Given a Morse function $f : \mathbb{CP}^n \rightarrow \mathbb{R}$, we consider the Floer equation with the Hamiltonian εf :

$$\frac{\partial u}{\partial \tau} + J_0 \frac{\partial u}{\partial t} = -\varepsilon \nabla f(u).$$

Let $U_0 \simeq \mathbb{C}^n$ be an affine chart, where

$$\begin{aligned} U_0 &= \{[z_0 : z_1 : \cdots : z_n] \mid z_i \in \mathbb{C}, z_0 \neq 0 \in \mathbb{C}\} \\ &= \{[1 : z_1 : \cdots : z_n] \mid z_i \in \mathbb{C}\}, \quad (i = 1, 2, \dots, n). \end{aligned}$$

Let

$$\chi : [-l, l] \rightarrow U_0 \simeq (\mathbb{C}^n, g_{FS})$$

be a segment of a Morse trajectory of f with end points $\chi(\pm l) = p_{\pm} \in \mathbb{C}^n$, and assume the full Morse trajectory extending χ to two critical endpoints is contained in the unit ball $B_1 = \{|z| \leq 1\} \subset U_0$. Let $u_{\pm} : (\mathbb{R} \times S^1, o_{\pm}) \rightarrow U_0 \simeq (\mathbb{C}^n, g_{FS})$ be two holomorphic curves,

$$u_{\pm}(\tau, t) = u(z) = A_{\pm} z^{\pm 1} + p_{\pm},$$

where A_{\pm} are two complex linearly independent vectors in \mathbb{C}^n , and

$$z = e^{2\pi(\tau + it)}, \quad o_{\pm} = \{-\pm \infty\} \times S^1.$$

We thus have the “disk-flow-disk” element (u_-, χ, u_+) in \mathbb{CP}^n with immersed joint points

$$p_{\pm} = \chi(\pm l) = u_{\pm}(o_{\pm}).$$

We can construct a family of maps $u_{\varepsilon} : \mathbb{R} \times S^1 \rightarrow \mathbb{C}^{n+1}$ approximately satisfies the above Floer equation when ε is small. Let

$$u_{\varepsilon}(\tau, t) = \varepsilon^{\alpha} e^{-2\pi \frac{t}{\varepsilon}} (A_+ z + A_- z^{-1}) + \chi(\varepsilon \tau)$$

with a fixed $\alpha > 0$. Then

$$\frac{\partial u_{\varepsilon}}{\partial \tau} + J_0 \frac{\partial u_{\varepsilon}}{\partial t} = \varepsilon \nabla f(\chi(\varepsilon \tau)) \approx \varepsilon \nabla f(u_{\varepsilon}(\tau, t)),$$

where $\chi(\varepsilon \tau) \approx u_{\varepsilon}(\tau, t)$ for $|\tau| \leq l/\varepsilon$ is justified by the following Proposition, and for $|\tau| > l/\varepsilon$, the following Proposition also proves $u_{\varepsilon} \approx \varepsilon^{\alpha} e^{-2\pi \frac{t}{\varepsilon}} (A_+ z + A_- z^{-1})$, hence

$$\frac{\partial u_{\varepsilon}}{\partial \tau} + J_0 \frac{\partial u_{\varepsilon}}{\partial t} \approx \left(\frac{\partial}{\partial \tau} + J_0 \frac{\partial}{\partial t} \right) \left[\varepsilon^{\alpha} e^{-2\pi \frac{t}{\varepsilon}} (A_+ z + A_- z^{-1}) \right] = 0 \approx \varepsilon \nabla f(u_{\varepsilon}(\tau, t)).$$

Although u_{ε} is only an approximate solution, it is very explicit and illustrates the mechanism of adiabatic convergence. It also indicates the convergence rates and L^2 -energy distribution of true solutions on different regions of $\mathbb{R} \times S^1$.

Proposition 16.1. *The adiabatic limit of $u_{\varepsilon}(\tau, t)$ as $\varepsilon \rightarrow 0$ is the “disk-flow-disk” configuration (u_-, χ, u_+) .*

Proof. We recall some useful inequalities of the Fubini-Study metric g_{FS} on the affine chart \mathbb{C}^n . Let g_{st} be the standard Euclidean metric on \mathbb{C}^n and the Euclidean norm be $|\cdot|$. Note that $g_{FS}(z) \leq \frac{1}{1+|z|^2} g_{st}(z) \leq g_{st}(z)$ for any $z \in \mathbb{C}^n$, so for any vectors $p, q \in \mathbb{C}^n$, we have

$$\begin{aligned} \text{dist}_{g_{FS}}(p, q) &\leq |p - q|, \\ \text{dist}_{g_{FS}}(p, q) &\leq 2 \frac{|p - q|}{|p|} \quad \text{if } |p - q| < \frac{|p|}{2}, \end{aligned}$$

where the second inequality is because both p and q are outside the Euclidean ball of radius $|p|/2$, and the Fubini-Study metric satisfies $g_{FS} \leq \frac{1}{|z|^2} g_{st}$.

Let $R(\varepsilon) = l/\varepsilon$. We first check that $d_H(u_\varepsilon(\tau, t)([-R(\varepsilon), R(\varepsilon)] \times S^1), \chi([-l, l])) \rightarrow 0$.

(i) For $|\tau| \leq R(\varepsilon) = l/\varepsilon$: we have

$$\begin{aligned} \text{dist}_{g_{FS}}(u_\varepsilon(\tau, t), \chi(\varepsilon\tau)) &\leq |u_\varepsilon(\tau, t) - \chi(\varepsilon\tau)| \\ &= \left| \varepsilon^\alpha e^{-2\pi \frac{l}{\varepsilon}} (A_+ z + A_- z^{-1}) \right| \\ &\leq \varepsilon^\alpha e^{-2\pi \frac{l}{\varepsilon}} \cdot (|A_+| e^{2\pi \frac{l}{\varepsilon}} + |A_-| e^{2\pi \frac{l}{\varepsilon}}) \\ &= \varepsilon^\alpha (|A_-| + |A_+|) \rightarrow 0 \end{aligned}$$

uniformly as $\varepsilon \rightarrow 0$.

Let

$$b(\varepsilon) = -\frac{1}{2\pi} \ln \varepsilon, \quad S(\varepsilon) = \alpha b(\varepsilon),$$

and the shift of u_\pm be

$$u_\pm^\varepsilon(\tau, t) := u_\pm(\tau - \pm l/\varepsilon - \alpha b(\varepsilon), t) = \varepsilon^\alpha e^{-2\pi \frac{l}{\varepsilon}} A_\pm z^{\pm 1} + p_\pm.$$

(ii) For $|\tau| \geq R(\varepsilon) = l/\varepsilon$: we consider two cases where $l/\varepsilon \leq |\tau| \leq l/\varepsilon + 2\alpha b(\varepsilon)$ or $|\tau| \geq l/\varepsilon + 2\alpha b(\varepsilon)$.

If $l/\varepsilon \leq \tau \leq l/\varepsilon + 2\alpha b(\varepsilon)$, then

$$\begin{aligned} \text{dist}_{g_{FS}}(u_\varepsilon(\tau, t), u_+^\varepsilon(\tau, t)) &\leq |u_\varepsilon(\tau, t) - u_+^\varepsilon(\tau, t)| \\ &\leq |\chi_\varepsilon(\tau) - p_+| + \left| \varepsilon^\alpha e^{-2\pi \frac{l}{\varepsilon}} A_- z^{-1} \right| \\ &\leq |\varepsilon \nabla f|_{C^0} \cdot |\tau - l/\varepsilon| + \varepsilon^\alpha |A_-| \\ &\leq 2 |\nabla f|_{C^0} \alpha \cdot \varepsilon b(\varepsilon) + \varepsilon^\alpha |A_-| \rightarrow 0 \end{aligned}$$

uniformly as $\varepsilon \rightarrow 0$.

If $\tau \geq l/\varepsilon + 2\alpha b(\varepsilon)$, we have

$$\begin{aligned} |u_+^\varepsilon(\tau, t)| &= \left| \varepsilon^\alpha e^{-2\pi \frac{l}{\varepsilon}} A_+ z \right| = |A_+| e^{2\pi(\tau - l/\varepsilon - \alpha b(\varepsilon))} \\ &\geq |A_+| e^{2\pi b(\varepsilon)\alpha} = |A_+| \varepsilon^{-\alpha} \rightarrow \infty, \\ |u_\varepsilon(\tau, t) - u_+^\varepsilon(\tau, t)| &= \left| \varepsilon^\alpha e^{-2\pi \frac{l}{\varepsilon}} A_- z^{-1} + \chi(\varepsilon\tau) \right| \leq \left| \varepsilon^\alpha e^{-2\pi \frac{l}{\varepsilon}} A_- z^{-1} \right| + |\chi(\varepsilon\tau)| \\ &\leq \varepsilon^\alpha |A_-| e^{-2\pi(l/\varepsilon + \tau)} + 1 \rightarrow 1 < \frac{|u_+^\varepsilon(\tau, t)|}{2}, \end{aligned}$$

when ε is small, hence

$$\begin{aligned}
\text{dist}_{g_{FS}}(u_\varepsilon(\tau, t), u_+^\varepsilon(\tau, t)) &\leq 2 \frac{|u_\varepsilon(\tau, t) - u_+^\varepsilon(\tau, t)|}{|u_\varepsilon(\tau, t)|} \\
&\leq 2 \frac{|\varepsilon^\alpha e^{-2\pi \frac{l}{\varepsilon}} A_- z^{-1} + \chi(\varepsilon \tau)| + |p_+|}{\left| \varepsilon^\alpha e^{-2\pi \frac{l}{\varepsilon}} A_+ z \right| - \left| \varepsilon^\alpha e^{-2\pi \frac{l}{\varepsilon}} A_- z^{-1} + \chi(\varepsilon \tau) \right|} \\
&\leq 2 \frac{(\varepsilon^\alpha |A_-| e^{-2\pi(l/\varepsilon + \tau)} + 1) + 1}{|A_+| \varepsilon^{-\alpha} - (\varepsilon^\alpha |A_-| e^{-2\pi(l/\varepsilon + \tau)} + 1)} \\
&\leq 2 \frac{3}{|A_+| \varepsilon^{-\alpha} - 2} \rightarrow 0
\end{aligned}$$

uniformly as $\varepsilon \rightarrow 0$. Combining these we have for $\tau \geq R(\varepsilon)$,

$$\text{dist}_{g_{FS}}(u_\varepsilon(\tau, t), u_+^\varepsilon(\tau, t)) \rightarrow 0,$$

or equivalently

$$\text{dist}_{g_{FS}}(u_\varepsilon(\tau + R(\varepsilon) + \alpha b(\varepsilon), t), u_+(\tau, t)) \rightarrow 0$$

uniformly as $\varepsilon \rightarrow 0$ on $[-\alpha b(\varepsilon), +\infty) \times S^1$, especially for any $[K, +\infty) \times S^1$ for any fixed $K \in \mathbb{R}$.

The case when $\tau \leq -l/\varepsilon$ is similar; we have

$$\text{dist}_{g_{FS}}(u_\varepsilon(\tau - R(\varepsilon) - \alpha b(\varepsilon), t), u_-(\tau, t)) \rightarrow 0$$

uniformly as $\varepsilon \rightarrow 0$ on $(-\infty, \alpha b(\varepsilon)] \times S^1$, especially for any $(-\infty, K] \times S^1$ for any fixed $K \in \mathbb{R}$.

Next we compute the energy $E(u_\varepsilon)$ on $\Theta_\varepsilon := [-R(\varepsilon), R(\varepsilon)] \times S^1$. We have

$$\begin{aligned}
\frac{\partial}{\partial \tau} u_\varepsilon(\tau, t) &= 2\pi \varepsilon^\alpha e^{-2\pi \frac{l}{\varepsilon}} \left(A_+ e^{2\pi(\tau + it)} - A_- e^{-2\pi(\tau + it)} \right) + \varepsilon \nabla f(\chi(\varepsilon \tau)) \\
\frac{\partial}{\partial t} u_\varepsilon(\tau, t) &= 2\pi i \varepsilon^\alpha e^{-2\pi \frac{l}{\varepsilon}} \left(A_+ e^{2\pi(\tau + it)} - A_- e^{-2\pi(\tau + it)} \right) \\
|du_\varepsilon(\tau, t)|_{g_{st}} &\leq C \varepsilon^\alpha e^{-2\pi \frac{l}{\varepsilon}} \cdot e^{2\pi|\tau|} + C\varepsilon.
\end{aligned}$$

For $|\tau| \leq R(\varepsilon)$, from the above third inequality we have

$$\begin{aligned}
\int_{[-R(\varepsilon), R(\varepsilon)] \times S^1} |du_\varepsilon|_{g_{FS}}^2 d\tau dt &\leq \int_{[-R(\varepsilon), R(\varepsilon)] \times S^1} |du_\varepsilon|_{g_{st}}^2 d\tau dt \\
&\leq 2C^2 \int_{[-R(\varepsilon), R(\varepsilon)] \times S^1} \left(\varepsilon^{2\alpha} e^{-4\pi(\frac{l}{\varepsilon} - |\tau|)} + \varepsilon^2 \right) d\tau dt \\
&= 2C^2 \left[\varepsilon^{2\alpha} \frac{1 - e^{-4\pi \frac{l}{\varepsilon}}}{2\pi} + 2l\varepsilon \right] \\
&\leq \tilde{C} (\varepsilon^{2\alpha} + \varepsilon) \rightarrow 0.
\end{aligned}$$

□

If the joint points of u_\pm are not immersed, in the next example we will see extra family of approximate solutions of the Floer equation beyond our pre-gluing construction. Let

$$u_\varepsilon(\tau, t) = \varepsilon^\alpha \left[e^{-2\pi k \frac{l}{\varepsilon}} A_+ z^k + e^{-2\pi m \frac{l}{\varepsilon}} A_- z^{-m} \right] + \beta(\varepsilon) P(z) + \chi(\varepsilon \tau),$$

where $k, m > 0$ are integers, and at least one of them > 1 , $P(z)$ is any Laurent polynomial of intermediate degree between z^k and z^{-m} with \mathbb{C}^n vector-valued coefficients, and $\beta(\varepsilon)$ is a fast vanishing real constant when $\varepsilon \rightarrow 0$. Then for fixed $l > 0$, similarly we can show u_ε has the adiabatic limit (u_-, χ, u_+) , where $u_\pm(z)$ are the holomorphic spheres in \mathbb{CP}^n that in the affine chart $U_0 \simeq \mathbb{C}^n$,

$$u_+(z) = A_+ z^k + p_+, \quad u_-(z) = A_- z^{-m} + p_-$$

for $z = e^{2\pi(\tau+it)}$. But other than the approximate solutions

$$u_\varepsilon(\tau, t) = \varepsilon^\alpha \left[e^{-2\pi k \frac{t}{\varepsilon}} A_+ z^k + e^{-2\pi m \frac{t}{\varepsilon}} A_- z^{-m} \right] + \chi(\varepsilon\tau)$$

of the Floer equation, now we have extra family of approximate solutions from the term $\beta(\varepsilon)P(z)$. This gives evidence that we can not prove surjectivity if there is non-immersed joint point.

The adiabatic degeneration with Lagrangian boundary condition can be constructed similarly, with the Lagrangian $L = \mathbb{RP}^n$ and $\chi(\tau)$ inside \mathbb{RP}^n , and $u_\varepsilon : \mathbb{R} \times [0, \frac{1}{2}] \rightarrow U_0 \simeq \mathbb{C}^n$,

$$u_\varepsilon(\tau, t) = \varepsilon^\alpha \left[e^{-2\pi k \frac{t}{\varepsilon}} A_+ z^k + e^{-2\pi m \frac{t}{\varepsilon}} A_- z^{-m} \right] + \beta(\varepsilon)P(z) + \chi(\varepsilon\tau)$$

be the approximate solution of Floer equation $\frac{\partial u}{\partial \tau} + J_0 \frac{\partial u}{\partial t} + \varepsilon \nabla f(u) = 0$ with Lagrangian boundary condition $u(\mathbb{R} \times \{0, \frac{1}{2}\}) \subset L$.

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